The Weyl Law of transmission eigenvalues and the completeness of the generalized transmission eigenfunctions without the complementing conditions

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Abstract

The transmission eigenvalue problem is a system of two second-order elliptic equations of two unknowns equipped with the Cauchy data on the boundary. In this talk we discuss the Weyl law for the eigenvalues and the completeness of the generalized eigenfunctions for the system without complementing conditions, i.e., the two equations of the system have the same coefficients for the second order terms. These coefficients are allowed to be anisotropic and are assumed to be of class C^2 only. This is based on a recent joint work with Hoai-Minh Nguyen.

Keywords: Transmission eigenvalue problem, inverse scattering, Weyl law, completeness, generalized eigenfunctions, Hilbert-Schmidt operators

1 Introduction

The transmission eigenvalue problem plays a role in the inverse scattering theory for inhomogeneous media. After four decades of extensive study, the spectral properties are known to depend on a type of contrasts of the media near the boundary. We refer the reader to [1] for a recent, and self-contained introduction to the transmission eigenvalue problem and its applications.

Natural and interesting questions on the interior transmission eigenvalue problem include: the *discreteness* of the spectrum, the *location* of transmission eigenvalues, the *Weyl law* of transmission eigenvalues and the *completeness* of the generalized eigenfunctions.

Let us describe the mathematical settings of the problem under consideration. Let Ω be a bounded, simply connected, open subset of $\mathbb{R}^d$ of class C^3 with $d \geq 2$. Let A_1, A_2 be symmetric, uniformly elliptic and continuous matrices in $\overline{\Omega}$ and let Σ_1, Σ_2 be two positive and continuous functions bounded from below by a positive constant.

A complex number $\lambda \in \mathbb{C}$ is called a transmission eigenvalue if there is a non-zero pair of functions $(u_1, u_2) \in [H^1(\Omega)]^2$ such that

$$\begin{cases} \operatorname{div}(A_1 \nabla u_1) - \lambda \Sigma_1 u_1 = 0 & \text{in } \Omega, \\ \operatorname{div}(A_2 \nabla u_2) - \lambda \Sigma_2 u_2 = 0 & \text{in } \Omega, \\ u_1 = u_2, \quad A_1 \nabla u_1 \cdot \nu = A_2 \nabla u_2 \cdot \nu & \text{on } \Gamma. \end{cases} \tag{1}$$

Here and in what follows $H^1(\Omega)$ denotes the standard Sobolev of functions in $L^2(\Omega)$ with derivatives belonging to $L^2(\Omega)$, Γ denotes $\partial\Omega$ and ν denotes the unit outward vector normal on Γ.

This problem has been studied under various conditions on the media (A_1, Σ_1) and (A_2, Σ_2). Recently, (Q.H.) Nguyen and (H.M.) Nguyen established the Weyl law of eigenvalues and the completeness of the generalized eigenfunctions under the complementing conditions adapted to this problem [3]. The complementing conditions for elliptic systems are originally due to Agmon et al. in the '60.

Concerning the degenerate case

$$A_1 = A_2 = A \text{ in } \Omega \tag{2}$$

and under the condition

$$\Sigma_1 \neq \Sigma_2 \text{ on } \partial\Omega \tag{3}$$

it was also shown by (Q.H.) Nguyen and (H.M.) Nguyen in [3] that the discreteness of the transmission eigenvalues holds. In this talk, we present recent results establishing the completeness of the generalized eigenfunctions and the Weyl law for the transmission eigenvalues under the additional assumption that for some $\Lambda > 0$

$$\|A\|_{C^2(\overline{\Omega})} + \|(\Sigma_1, \Sigma_2)\|_{C^1(\overline{\Omega})} \leq \Lambda. \tag{4}$$

Previous results concerning the transmission eigenvalue problem under the condition (2) were obtained in [5].

2 Main results

Let $(\lambda_j)_j$ be the set of transmission eigenvalues associated with the transmission eigenvalue problem (1).

Theorem 1 (Fornerod & Nguyen [2]) *Assume that* (2),(3) *and* (4) *hold. Let* $\mathcal{N}(t)$ *denote the counting function i.e.*

$$\mathcal{N}(t) = \#\{j \in \mathbb{N} : |\lambda_j| \le t\}.$$

Then

$$\mathcal{N}(t) = \mathbf{c}t^{\frac{d}{2}} + o(t^{\frac{d}{2}}) \text{ as } t \to +\infty,$$

where

$$\mathbf{c} := \frac{1}{(2\pi)^d} \sum_{\ell=1}^{2} \int_{\Omega} \left| \left\{ \xi \in \mathbb{R}^d ; \langle A_\ell(x)\xi, \xi \rangle < \Sigma_\ell(x) \right\} \right| dx.$$

Concerning the completeness of the generalized eigenfunction we have

Theorem 2 (Fornerod & Nguyen [2]) *Assume that* (2),(3) *and* (4) *hold. The set of generalized eigenfunction pairs associated to the problem* (1) *is complete in* $L^2(\Omega) \times L^2(\Omega)$.

The analysis is based on the well-posedness and the regularity of the following Cauchy problem. Let $\gamma > 0$. There exists $\lambda_0 > 0$ such that for every $\lambda \in \mathbb{C}$ with $|\Im(\lambda)| \ge \gamma|\lambda|$ and $|\lambda| > \lambda_0$ for every $(f_1, f_2) \in [L^2(\Omega)]^2$ there exists a unique solution $(u_1, u_2) \in [L^2(\Omega)]^2$ with $u_1 - u_2 \in H^2(\Omega)$ of

$$\left\{ \begin{array}{rll} \operatorname{div}(A\nabla u_1) - \lambda\Sigma_1 u_1 & = f_1 & \text{in } \Omega, \\ \operatorname{div}(A\nabla u_2) - \lambda\Sigma_2 u_2 & = f_2 & \text{in } \Omega, \\ u_1 - u_2 = A\nabla(u_1 - u_2)\cdot\nu & = 0 & \text{on } \Gamma. \end{array} \right. \tag{5}$$

It is worth noting that the Cauchy problem (5) is degenerate due to the condition (2). One of the key step in order to overcome this difficulty is to derive a priori estimate in a half space. Due to (2) the situation is non-standard and the analysis requires the stability of the Cauchy problem with regularity on the data up to the second derivatives. The regularity of the solution (u_1, u_2) of (5) can be improved only by improving the one of the difference $f_1 - f_2$. This is distinct from the complementing conditions on (A_1, Σ_1) and (A_2, Σ_2) considered in [4].

In order to obtain Theorem 1 and Theorem 2 we associate an operator to a system which is equivalent to (5). This procedure is standard in the literature and has been initiated in [5]. Then, we deduce that a certain power of this operator is in fact a Hilbert-Schmidt operator. This allows to obtain Theorem 1 and Theorem 2 by using the theory of Hilbert-Schmidt operators. We follow the approach of [4], but with extra observations and new ideas to solve the problems due to the degenerate condition (2).

3 Overline of the talk

In this talk we will present the proofs of Theorem 1 and Theorem 2. This will be divided two parts. The first one is devoted to the study of the Cauchy problem associated with the interior transmission problem. Precisely, we will focus on the existence, the uniqueness and the regularity of the solution to the Cauchy problem (5), with a particular care on the new material used in the proof of these results. The second part is devoted to Theorem 1 and Theorem 2.

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