dGFEM-BEM mortar coupling for the Helmholtz problem in three dimensions

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Abstract

We present a way of coupling an interior penalty discontinuous Galerkin method with a boundary element method for the Helmholtz equation in 3d. The coupling is realized with a mortar variable related to an impedance trace. We prove quasi-optimality of the h- and p-versions of the scheme, under a threshold condition on the approximation properties of the discrete spaces.

Keywords: discontinuous Galerkin method; boundary element method; mortar coupling; Helmholtz equation;

1 Model Problem

Consider a bounded domain $\Omega \subseteq \mathbb{R}^3$ with analytic boundary Γ. Fix a characteristic wave speed $k > 0$ and let $n : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function with $n(x) > 0$ and $n(x) = 1$ outside of Ω. Assume that the right-hand side f is analytic and satisfies $\operatorname{supp}(f) \subseteq \Omega$.

We aim to approximate solutions to the Helmholtz problem

$$-\Delta u - (kn)^2 u = f \text{ in } \mathbb{R}^3 \tag{1a}$$

$$\lim_{|x|\to+\infty} |x|(\partial_{|x|} u - iku) = 0. \tag{1b}$$

Most of the details to this talk can be found in the preprint [1].

2 Mortar coupling

We introduce two auxiliary variables on the boundary $m := \partial_n^- u + ik\gamma^- u$ and $u^{ext} := \gamma^+ u$ where γ^- and ∂_n^- is the interior trace and normal derivative. γ^+ correspondingly is the exterior trace. This gives the coupled problem:

$$\begin{cases} -\Delta u - (kn)^2 u = f & \text{in } \Omega, \\ \partial_{\mathbf{n}_\Gamma} u + iku = m & \text{on } \Gamma, \\ u^{ext} = \mathcal{P}_{ItD} m & \\ \gamma^- u = \left(\frac{1}{2} + \mathcal{K}_k\right) u^{ext} - \mathcal{V}_k(m - iku^{ext}). & \end{cases} \tag{2}$$

For discretization, we use an interior penalty dG discretization [3] for the equation posed on Ω based on a piecewise polynomial space V_h. For the boundary integrals, we use a combined-field type representation for the Impedance-to-Dirichlet operator and BEM spaces $W_h \subseteq H^{-1/2}(\Gamma)$ and $Z_h \subseteq H^{1/2}(\Gamma)$ for the approximation of m and u^{ext}, respectively.

3 Analysis

We use two different norms:

$$\begin{aligned} \|(v,\lambda,v^{ext})\|_{\mathrm{dG}}^2 &:= \|\nabla_h v\|_{0,\Omega}^2 + \|k\, v\|_{0,\Omega}^2 \\ &+ k^{-1}\|\beta^{1/2}[\![\nabla_h v]\!]\|_{0,\mathcal{F}_h^I}^2 + k\|\alpha^{1/2}[\![v]\!]\|_{0,\mathcal{F}_h^I}^2 \\ &+ k^{-1}\|\delta^{1/2}\nabla_h v\cdot\mathbf{n}_\Gamma\|_{0,\Gamma}^2 + k\|v\|_{0,\Gamma}^2 \\ &+ \|\lambda\|_{-\frac{1}{2},\Gamma}^2 + \|v^{ext}\|_{\frac{1}{2},\Gamma}^2 \end{aligned}$$

and

$$\begin{aligned} \|(v,\lambda,v^{ext})\|_{\mathrm{dG}^+}^2 &:= \|(v,m,u^{ext})\|_{\mathrm{dG}}^2 \\ &+ k^{-1}\|\alpha^{-1/2}\{\!\{\nabla_h v\}\!\}\|_{0,\mathcal{F}_h^I}^2 + \|h^{1/2}p^{-1}\,\lambda\|_{0,\Gamma}^2 \end{aligned}$$

where $\|\cdot\|_{0,\mathcal{F}_h^I}^2$ denotes the sum of the L^2-norms over all interior facets, $[\![\cdot]\!]$ and $\{\!\{\cdot\}\!\}$ denote the jump and mean across facets respectively. α, β and δ are stabilization parameters that need to be chosen appropriately.

We denote the sesquilinear form obtained by Galerkin discretization of (2) by $\mathcal{T}(\cdot,\cdot)$. This form satisfies a Gårding inequality in the dG-norm, i.e., there exists $\varepsilon > 0$, $c > 0$ such that

$$\begin{aligned} &(\mathrm{Re} + \varepsilon\,\mathrm{Im})(\mathcal{T}((v,\lambda,v^{ext}),(v,\lambda,v^{ext}))) \\ &\qquad \geq c\|(v,\lambda,v^{ext})\|_{\mathrm{dG}}^2 - \Theta(v,\lambda,v^{ext}), \end{aligned}$$

with $\Theta(v,\lambda,v^{ext})$ a compact perturbation. Stability holds in the stronger dG$^+$-norm. By carefully designing the discretization scheme, we can ensure adjoint consistency, i.e., the transpose of the discretization matrix is the dG-discretization of a natural adjoint problem to (2). This enables the use of the powerful Schatz argument.

4 Reconstruction operator

The main obstacle when trying to prove Gårding inequality and stability of the bilinear form is that while the dG approximation u_h in the interior has jumps across the faces, the boundary integral operators require functions which are $H^{1/2}(\Gamma)$-conforming. Our solution: a novel reconstruction operator $\mathcal{P}$ in the spirit of [2], which maps piecewise H^1 functions to globally continuous functions.

Theorem 1 *Let $\mathcal{T}$ be a shape-regular mesh of size h on Ω. Then, for each $p \in \mathbb{N}$ there exists a linear operator $\mathcal{P} : H^1_{\mathrm{pw}}(\Omega_h) \to H^1(\Omega)$ that satisfies, for all $v \in H^1_{\mathrm{pw}}(\Omega_h)$,*

$$\begin{aligned}\|\nabla \mathcal{P} v\|_{0,\Omega} &\lesssim \|\nabla_h v\|_{0,\Omega} + \|h^{-1/2} p [\![v]\!]\|_{0,\mathcal{F}_h^I},\\ \|\mathcal{P} v\|_{0,\Omega} &\lesssim \|h p^{-2} \nabla_h v\|_{0,\Omega}\\ &\quad + \|v\|_{0,\Omega} + \|h^{1/2} p^{-1} [\![v]\!]\|_{0,\mathcal{F}_h^I},\end{aligned}$$

as well as the approximation property

$$\begin{aligned}&\|h^{-1/2} p\,(I - \mathcal{P}) v\|_{0,\Gamma}\\ &\qquad \lesssim \|\nabla_h v\|_{0,\Omega} + \|h^{-1/2} p [\![v]\!]\|_{0,\mathcal{F}_h^I}.\end{aligned}$$

The construction of Theorem 1 is such that we first use a quasi-interpolation into piecewise linear functions on an artificial mesh of size $\mathcal{O}(hp^{-2})$. There, we use the "averaging of degrees of freedom" operator from [2]. This gives **robust stability and approximation estimates for piecewise polynomials of degree** p. The price is that the operator does not map into the space Z_h of piecewise polynomials on the original triangulation. Instead, it maps to the space of piecewise linears on an artificial refined grid.

5 Main Theorem

Using all these ingredients, we can derive a quasi-optimality result.

Theorem 2 *Let the solution (u, m, u^{ext}) to (2) be in $H^{\frac{3}{2}+t}(\Omega) \times L^2(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ for some $t > 0$, and let $(u_h, m_h, u_h^{ext}) \in V_h \times W_h \times Z_h$ be the discrete solution. Assume that the adjoint problem can be approximated sufficiently well. Then:*

$$\begin{aligned}&\left|\!\left|\!\left| u - u_h, m - m_h, u^{ext} - u_h^{ext} \right|\!\right|\!\right|_{dG}\\ &\lesssim \inf_{v_h, \lambda_h, v_h^{ext}} \left|\!\left|\!\left| u - v_h, m - \lambda_h, u^{ext} - v_h^{ext} \right|\!\right|\!\right|_{dG^+},\end{aligned}$$

where the infimum is taken over all $(v_h, \lambda_h, v_h^{ext}) \in V_h \times W_h \times Z_h$.

The analysis of [1] is not explicit in k. We will briefly discuss how to address this issue.

6 Numerical results

We performed numerical simulations in which we compared the numerical approximations of our scheme to a known exact smooth solution. Namely, we used the domain $\Omega := (-1, 1)^3$, the coefficient $n = 1$ and set

$$u(x, y, z) := \begin{cases} \sin(k\,x)\cos(ky) & (x, y, z) \in \Omega \\ \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}} & \text{otherwise.} \end{cases}$$

Note that this solution is discontinuous across Γ, and thus does not strictly fit (1). One can easily modify the right-hand sides of the scheme to account for given jumps across Γ.

We observe in Figure 1 that the method performs as expected when refining the mesh, i.e. the error is $\mathcal{O}(h^p)$ where p denotes the order of the polynomials employed.

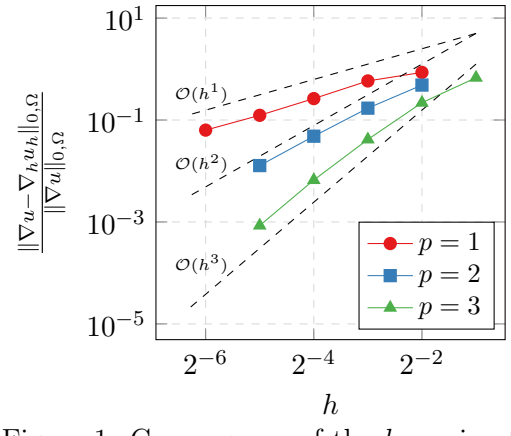

Figure 1: Convergence of the h-version for $k = 2\sqrt{3}\pi$.

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