Solution of a non-linear eigenvalue problem for Photonic Crystal Fiber applications

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Abstract

A method to solve a non-linear eigenvalue problem coming from boundary integral equations is studied. The equations are provided by photonic crystal fiber applications.

Keywords: Boundary Element Method, Photonic Crystal Fiber, Non-linear eigenvalue problem

1 Introduction

Photonic crystal fibers (PFC) are systems that have been widely used for decades to allow light propagation. The geometry, the dielectric characteristics of the materials and the wavelength of the source are the main parameters to determine the effective refractive index of the medium. The complexity of these optical systems – heterogeneous structure, geometry of the section and the micrometric order of magnitude – makes the numerical methods mandatory to quickly design a PCF for the desired application. The finite-difference time-domain method and the finite element method are common approaches used to solve the problem of propagation in a PCF. However, these methods may require a huge amount of memory and computation time, according to the size of the mesh for a PCF with several inclusions. Solutions based on the Boundary Element Method (BEM) [1] have been proposed to reduce the size of the problem. They allow to consider only the mesh on the boundary of the inclusions but are limited by the resolution of a nonlinear eigenvalue problem. Usually solved by Müller's method, it requires a rather precise knowledge of the solution as a starting point. We propose as an alternative a search method based on contour integrals and rational interpolation [2] not limited by these difficulties.

2 Boundary Integral Equations

We apply the formulation proposed in [1] but other formulations could be considered. Using the free space Green function

$$G(\mathbf{r},\tilde{\mathbf{r}}) = \frac{j}{4} H_0^{(1)}\left((k^2-\beta^2)|\mathbf{r}-\tilde{\mathbf{r}}|\right),\ \mathbf{r}\neq\tilde{\mathbf{r}} \quad (1)$$

where $H_0^{(1)}$ is the Hankel function. Magnetic field components $u = H_x$ or H_y can be represented for $\mathbf{r} \notin \partial\Omega_j$ by

$$\begin{aligned} u(\mathbf{r}) = &\int_{\partial\Omega_j} G(\mathbf{r},\tilde{\mathbf{r}}) \frac{\partial u(\tilde{\mathbf{r}})}{\partial \boldsymbol{\nu}} \mathrm{d}s(\tilde{\mathbf{r}}) \\ &- \int_{\partial\Omega_j} \frac{\partial G(\mathbf{r},\tilde{\mathbf{r}})}{\partial \boldsymbol{\nu}(\tilde{\mathbf{r}})} u(\tilde{\mathbf{r}}) \mathrm{d}s(\tilde{\mathbf{r}}), \end{aligned} \quad (2)$$

where Ω_j is an homogeneous inclusion. Expressing continuity conditions on an interface $\partial\Omega_j$

$$[\partial_\nu H_\nu] = [E_z] = 0$$

and using Maxwell's equations

$$ik_0 n^2 E_z = \partial_y H_x - \partial_x H_y,$$

the system can be expressed under the form

$$\mathcal{M} \begin{bmatrix} H_x \\ H_y \end{bmatrix} = 0$$

with

$$\mathcal{M} = \begin{bmatrix} \frac{1}{n^2}\left(\nu_y\partial_\nu + \nu_x\partial_\tau\right) & \frac{1}{n^2}\left(\nu_x\partial_\nu - \nu_y\partial_\tau\right) \\ \nu_x\partial_\nu & \nu_y\partial_y \end{bmatrix}.$$

It leads to solve a non-linear eigenvalue problem

$$F(n_{\mathrm{eff}})\boldsymbol{\mu}^{\boldsymbol{H}} = 0. \quad (3)$$

Matrix F, coming from continuous operator $\mathcal{M}$, depends on the effective refractive index n_{eff}.

3 Algorithm for solving problem (3)

For solving a non-linear eigenvalue problem as (3), approaches exploiting contour integrals have been introduced in [2] and some improved variants have been proposed in several references (see for instance [3]). It consists to numerically compute integrals

$$\frac{1}{2\pi i}\int_{\mathcal{C}} f(z)F(z)^{-1}\hat{V}dl$$

where $\mathcal{C}$ is a smooth contour enclosing the eigenvalues of interest, f an analytic function and $\hat{V}$

a random matrix of L columns. Keldysh' theorem provides us a link with the eigenvalues and right/left eigenvectors; for instance in the case of simple eigenvalues, we can write

$$F(z)^{-1} = \sum_{i=1}^{n(\mathcal{C})} v_k w_k^H \frac{1}{z - \lambda_k} + R(z)$$

where $n(\mathcal{C})$ is the number of eigenvalues inside $\mathcal{C}$, λ_k are these eigenvalues, v_k and w_k the corresponding right and left eigenvectors and R is an analytic function. Computing contour integrals then enables us to focus only on the contribution of the rational part containing information on the eigenvalues. In most of the proposed algorithms, f is chosen as the moments z^k, with $k = 1, \ldots, K$ and it enables to convert the nonlinear eigenvalue problem into a linear generalized eigenvalue problem.

Connections have also been done between this approach and rational interpolation; it leads in [3] to an algorithm called SS-RI (Sakurai-Sugiura method with Rational Interpolation). The proposed modifications enable to provide more numerical stability, in particular for large K, and also to choose "quadrature" points not only on the contour $\mathcal{C}$ but also inside. This approach is considered for the numerical results.

4 Numerical Results

The tests in this section correspond to several discretisations of the fiber shown Figure 1 with the following physical parameters : free space wavelength is $\lambda = 1.45\mu m$, each hole has a diameter of $5\mu m$ and is discretized by $N = 40$ unknowns. The refractive index of the glass matrix is 1.45 and the medium surrounding the hole is infinite. We have tested several integral contour proposed in [3] but the most convenient is to keep a segment $[a_x, b_x]$ on the real axis discretized with N_x Chebyshev points. It is then interesting to note that without any computation outside the real axis, the method gives a good approximate value of the imaginary part. Table 1 gives zero(s) obtained for several discretizations and lengths of the segment. As expected, more Chebyshev points are necessary when the size of the segment increases. The main result is that a good approximation can be obtained with few computations. To start the search, a rough interval scan have to be performed and a few Müller iterations are useful to refine the solution. We point out that the algorithm may give several close solutions (two in our example) and that the refinement shows that they are equal. The SS-RI algorithm requires less computations than the brute force Müller algorithm [1] and does not require a very close initial guess. It makes the method more relevant to compute new PCF.

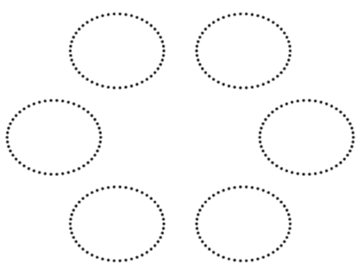

Figure 1: PCF hexapole for numerical tests

[ax,bx]	N_x	n_{eff}
[1.4453,1.4454]	3	(1.4453952345,3.21264e-08)
[1.4453,1.4454]	5	(1.4453952341,3.19389e-08)
[1.4453,1.4454]	10	(1.4453952341,3.19388e-08)
[ax,bx]	N_x	n_{eff}
[1.445,1.446]	5	(1.4453952335,2.859892e-08)
[1.445,1.446]	10	(1.4453952341,3.19371e-08)
[1.445,1.446]	20	(1.4453952341,3.19393e-08)
[ax,bx]	N_x	n_{eff}
[1.44,1.45]	20	(1.4453952506,4.34332e-08)
[1.44,1.45]	50	(1.4453952304,3.997883e-08)
[1.44,1.45]	100	(1.4453952343,3.1748179e-08)

Table 1: SS-RI algorithm Results

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