A new theory for acoustic transmission problems with variable coefficients modeled as stable integral equations

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Abstract

In this paper we consider transmission problems in either the full space or a subset and solve the heterogeneous Helmholtz equation. By using tools from harmonic analysis we construct the layer potentials in an abstract way, as solution of certain transmission problem, not using a representation via Green's functions. We derive a formulation as boundary integral equations, which is coercive, self-dual and continuous, hence well posed.

Keywords: Boundary element methods, layer potentials, frequency-domain wave equation

1 Introduction

We consider the domain $\Omega \subseteq \mathbb{R}^3$ with (possibly empty) boundary Γ; it is partitioned in n_Ω subdomains $\Omega_j, 1 \le j \le n_\Omega$. We define the *skeleton* $\Sigma := \bigcup_{1\le j\le n_\Omega} \Gamma_j$. We analyze the *Helmholtz transmission problem*: for $\operatorname{Re} s > 0$

$$\begin{cases} p^2 s^2 u - \operatorname{div}(A\nabla u) = 0 \text{ in } \Omega_j, 1 \le j \le n_\Omega, \\ \qquad [u]_{\mathrm{D};j} = 0 \text{ on } \Gamma_j, 1 \le j \le n_\Omega, \\ \qquad [u]_{\mathrm{N};j} = g_j \text{ on } \Gamma_j, 1 \le j \le n_\Omega, \\ \text{Boundary conditions on } \Gamma, \\ \text{Radiation conditions if } \Omega \text{ is unbounded,} \end{cases}$$

where $[\cdot]_{\mathrm{D};j}[\cdot]_{\mathrm{N};j}$ denote the jump and co-normal jump at $\Gamma_j := \partial\Omega_j$.

For simplicity g_j is 0, except possibly on some closed boundary Γ_j.

We allow for general coefficients A, p, i.e:

$$A \in L^\infty\left(\Omega, \mathbb{R}^{3\times 3}_{\mathrm{sym}}\right), \quad p \in L^\infty(\Omega, \mathbb{R});$$

moreover it is assumed that p is positive, and A is uniformly positive definite.

2 Layer Potentials

The layer potentials are defined for each subdomain Ω_j separately. They are introduced in an abstract way, by PDE techniques, and not relying on the Green's function [2].

For this goal we consider the subdomain Ω_j with boundary $\Gamma_j := \partial\Omega_j$, and extend the coefficients $p|_{\Omega_j}$, $A|_{\Omega_j}$ to positive (definite) L^∞ coefficients p_j, A_j defined on the full space $\mathbb{R}^3$.

We introduce the sesquilinear form

$$\ell_j(s)\colon H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \to \mathbb{C}$$
$$\ell_j(s)(u,v) := \left\langle p_j^2 s^2 u, \overline{v}\right\rangle_{\mathbb{R}^3} + \left\langle A_j \nabla u, \overline{\nabla v}\right\rangle_{\mathbb{R}^3},$$

with $\mathsf{L}_j(s)\colon H^1(\mathbb{R}^3) \to H^{-1}(\mathbb{R}^3)$ being the associated operator.

The solution operator $\mathcal{N}_j(s)$ (called sometimes acoustic Newton potential) for this sesquilinear form is given by

$$\ell_j(s)(\mathcal{N}_j(s)f, w) = \langle f, \overline{w}\rangle_{\mathbb{R}^3} \quad \forall w \in H^1(\mathbb{R}^3).$$

The single layer potential is defined by

$$\mathsf{S}_j(s)\phi := s^{1/2}\mathcal{N}_j(s){\gamma_{\mathrm{D};j}}'\phi,$$

where $\gamma_{\mathrm{D};j}$ is the usual trace operator and ${\gamma_{\mathrm{D};j}}'$ is its dual. For the double layer, given any $\phi \in H^{1/2}(\Gamma_j)$ consider a function $f \in H^1(\mathbb{R}^3)$ such that $\gamma_{\mathrm{D};j} f = s^{-1/2}\phi$; then

$$\mathsf{D}_j(s)\phi|_{\Omega_j} := -f|_{\Omega_j} + \left(\mathcal{N}_j(s)\mathsf{L}_j(s)\, f|_{\Omega_j}\right)\Big|_{\Omega_j};$$
$$\mathsf{D}_j(s)\phi|_{\Omega_j^c} := f|_{\Omega_j^c} - \left(\mathcal{N}_j(s)\mathsf{L}_j(s)\, f|_{\Omega_j^c}\right)\Big|_{\Omega_j^c}.$$

The definition does not depend on f, only on its trace. Note that these s-dependent scalings will lead to boundary operators which allow for norm estimates with optimal scaling in terms of the frequency s.

These operators satisfy norm estimates consistent with [6]. We refer to [4] for the details.

3 Skeleton Operators

The usual skeleton operators are obtained from the layer potentials:

$$\mathsf{V}_j(s)\colon H^{-1/2}(\Gamma_j) \to H^{1/2}(\Gamma_j),$$
$$\mathsf{K}_j(s)\colon H^{1/2}(\Gamma_j) \to H^{1/2}(\Gamma_j),$$
$$\mathsf{K}'_j(s)\colon H^{-1/2}(\Gamma_j) \to H^{-1/2}(\Gamma_j),$$
$$\mathsf{W}_j(s)\colon H^{1/2}(\Gamma_j) \to H^{-1/2}(\Gamma_j),$$

$$\begin{aligned}\mathsf{V}_j(s) &:= \{\!\{\mathsf{S}_j(s)\}\!\}_{\mathrm{D};j},\\ \mathsf{K}_j(s) &:= \{\!\{\mathsf{D}_j(s)\}\!\}_{\mathrm{D};j},\\ \mathsf{K}'_j(s) &:= \{\!\{\mathsf{S}_j(s)\}\!\}_{\mathrm{N};j},\\ \mathsf{W}_j(s) &:= -\{\!\{\mathsf{D}_j(s)\}\!\}_{\mathrm{N};j},\end{aligned}$$

where $\{\!\{\cdot\}\!\}_{\mathrm{D};j}, \{\!\{\cdot\}\!\}_{\mathrm{N};j}$ denote the mean of the Dirichlet and Neumann traces.

The Calderón operators are first constructed for a single domain in the usual way:

$$\mathbf{C}_j(s) := \begin{bmatrix} -\mathsf{K}_j(s) & \mathsf{V}_j(s) \\ \mathsf{W}_j(s) & \mathsf{K}_j(s)' \end{bmatrix} - \frac{Id}{2};$$

they are then collected in the block-diagonal operator $\mathbf{C}(s) := \operatorname{diag}_{1\le j\le n_\Omega} \mathbf{C}_j(s)$ having the trace data on the full skeleton as domain.

We incorporate homogeneous transmission conditions into the function space where the solution is sought by using the *single trace* space. Homogeneous boundary conditions are also incorporated in the function space. Inhomogeneities are treated with suitable offset functions, which modify the right-hand side. We refer to [3, 5] for details.

At this point we are able to prove our main theorem, namely the stability of the sesquilinear form associated to the boundary integral operators.

Theorem 1. *The operator $\mathbf{C}(s)$ is continuous. Let $\sigma := \frac{s}{|s|}$, and denote with $\langle \cdot, \bar{\cdot} \rangle_{\boldsymbol{X}}$ the natural scalar product in the product trace space; then the following coercivity estimate holds for all $\boldsymbol{\Phi}$ in the single trace space which satisfy the homogeneous boundary conditions on Γ:*

$$\operatorname{Re} \left\langle \mathbf{C}\boldsymbol{\Phi}, \overline{\sigma\boldsymbol{\Phi}} \right\rangle_{\boldsymbol{X}} \ge C \min\{1, |s|^2\} \frac{\operatorname{Re} s}{|s|^2} \|\boldsymbol{\Phi}\|^2_{\boldsymbol{X}},$$

for some $C > 0$ which does not depend on s.

It follows that the resulting problem in weak form is well-posed [1, 3, 4].

From the implementation point of view the single trace space is enforced by only keeping one pair of traces at each interface, adding the contributions from both domains in order to apply the discretized operator.

4 Conclusions

- We present a general method to transform acoustic transmission problems with mixed boundary conditions to a system of non-local Calderón operators on the skeleton, without relying on the explicit knowledge of the Green's function. The resulting skeleton operators are coercive, self-dual and continuous.
- If the Green's function is explicitly known, the Calderón operators admit a representation as skeleton integrals, which allow for a standard discretization by conforming Galerkin BEM.

References

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