Localization landscape for interacting Bose gases in one-dimensional speckle potentials

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Abstract

Using the localization-landscape theory [1], we investigate the properties and the shape of the ground state (GS) of a gas of ultracold bosons in one-dimensional (1D) speckle potentials, starting from the Gross-Pitaevskii equation (GPE). For attractive interactions, we find approximate relations holding between the localization length and the disorder parameter as well as between the former quantity and the nonlinear coefficient. For weakly repulsive interactions, we prove that the ground state ψ_0 of the GPE can be understood as a superposition of a finite number of single-particle (SP) states. We show numerically that, for intermediate repulsive interactions, ψ_0 follows the modulations of the effective potential. We further prove that, for given parameters of the SP Hamiltonian, there exists a value of the nonlinear coefficient at which ψ_0 is well predicted by the normalized localization landscape.

Keywords: Nonlinear waves, Atomic gases, Random and disordered media

1 Introduction

We consider ultracold dilute Bose gases in a geometry with a 1D random potential along the x axis and a two-dimensional harmonic potential with frequency $\omega_\perp$ in the (y, z) plane. The level spacing between two neighboring eigenstates of the SP 1D problem along x is assumed to be smaller than zero-point energy $E_\perp := \hbar\omega_\perp$ of the harmonic oscillator (HO), so the GS can be factorized into the one of the two-dimensional (2D) HO and that of a one-dimensional GPE. After integrating out the transverse-plane wave-functions, the GPE along the x axis reads [2]:

$$\Bigg[\underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_R(x) + \hbar\omega_\perp}_{=:H^{\mathrm{sp}}} + \\ + \underbrace{2\hbar\omega_\perp a_s}_{=:g}(N-1)|\psi_0(x)|^2\Bigg]\psi_0(x) = \mu\psi_0(x) , \quad (1)$$

where V_R denotes the random potential, H^{sp} the SP Hamiltonian, g labels the nonlinear coupling and a_s represents the s-wave scattering length. The random potential treated here is a correlated speckle potential, whose probability distribution in the space obeys the Rayleigh law,

$$P(V_R) = \frac{\Theta_H(V_R V_0)}{V_0}\mathrm{e}^{-V_R/V_0} , \quad (2)$$

where $\Theta_H(x)$ is Heaviside's step function and V_0 the disorder parameter. The spatial correlation profile $C(x)$ of the potential is chosen to be Gaussian:

$$C(x) := \overline{[V_R(0) - V_0]\,[V_R(x) - V_0]} = \\ = V_0^2\mathrm{e}^{-\frac{x^2}{2\sigma^2}} , \quad (3)$$

in which σ denotes the correlation length, whereas $\bar{\cdot}$ indicates the average over the configurations of the disordered potential.

2 The localization landscape

From the one-dimensional problem found in Eq. (1), a localization landscape (LL) function $u(x)$ can be defined in analogy to the one introduced in Ref. [1],

$$H^{\mathrm{sp}}(x)u(x) = 1 , \quad (4)$$

satisfying the boundary conditions $u(x)|_{x=\pm\frac{L}{2}} = 0$, where L is the length of the 1D domain. Similarly to the procedure carried out in Ref. [1], the solution ψ can be expressed as a product of the landscape function u and the auxiliary function φ. Equation (1) can be thus recast as:

$$-\frac{\hbar^2}{2m}\left[\frac{1}{u^2}\frac{\partial}{\partial x}\left(u^2\frac{\partial\varphi}{\partial x}\right)\right] + \frac{\varphi}{u} + \\ + g(N-1)|u|^2|\varphi|^2\varphi = \mu\varphi , \quad (5)$$

which is a GP-like equation with a different elliptic differential operator and a space-dependent coefficient of the nonlinear interaction term. The effective random potential is now given by $W(x) := u(x)^{-1}$, as proven in the SP case [3].

3 Attractive interactions

We numerically computed ground state ψ_0 of the GPE using the split-step Crank-Nicolson method [4], which is based on imaginary-time evolution. In the case of attractive interactions, the wavefunction is localized and its asymptotical behaviour can be well approximated as:

$$\psi_0(x) \approx \sqrt{\frac{2}{\lambda_L + \lambda_R}}\begin{cases}\mathrm{e}^{\frac{(x-x_0)}{\lambda_L}} & x < x_0 \\ 1 & x = x_0 \\ \mathrm{e}^{\frac{-(x-x_0)}{\lambda_R}} & x > x_0\end{cases} , \quad (6)$$

where λ_L and λ_R represent the left and the right localization lengths respectively. The latter quantities appear to shrink as $|g(N-1)|$ is increased. By computing the localization lengths for variable nonlinear coefficients and performing nonlinear regressions, we find that the following relations hold:

$$\tilde{\lambda}_n = \frac{a_n}{\tilde{g}(N-1)+b_n} + c_n\,, \qquad (7)$$

where $\tilde{\lambda}_n := \lambda_n/l_\perp$ with $l_\perp := \sqrt{\frac{\hbar}{m\omega_\perp}}$ and $n = L, R$, whereas $\tilde{g} = g/(E_\perp l_\perp)$. By proceeding with an analogous method, we numerically assessed the validity of the following relation with the disorder mean value:

$$\tilde{\lambda}_n = \frac{A_n}{(\tilde{V}_0+B_n)^2} + C_n\,, \qquad (8)$$

where $\tilde{V}_0 := V_0/E_\perp$, which is also consistent with the known result in the absence of interactions [5].

4 Repulsive interactions

For weak repulsive interactions ($\mu \sim E_0$), we prove that $\psi_0(x)$ can be expressed as a superposition of the N_s lowest-lying SP eigenstates of H^{sp}:

$$\psi_0^{\mathrm{lcsp}}(x) = \sum_{j=0}^{N_s-1} c_j \psi_j^{\mathrm{sp}}(x)\,, \qquad (9)$$

where the coefficients $\{c_j\}$ must satisfy $\sum_{j=0}^{N_s-1}|c_j|^2 = 1$. N_s can be reckoned as $N_s \approx n^{\mathrm{sp}}\big(E_0^{(0)}\big)$, which is the number of SP states whose energy lies below $\tilde{E}_0^{(0)}$, i.e. the integrated density of states (IDoS) n^{sp} evaluated at $\tilde{E}_0^{(0)}$. The latter quantity is defined as:

$$E_0^{(0)} := E_0^{\mathrm{sp}} + \frac{g(N-1)}{2}\int_{-L/2}^{L/2} |\psi_0^{\mathrm{sp}}(x)|^4\,\mathrm{d}x\,. \qquad (10)$$

In the presence of repulsive interactions, in contrast with the attractive case, the wavefunction $\psi_0(x)$ gets increasingly delocalized (with decreasing oscillation amplitude) as the nonlinearity coefficient is raised. For intermediate repulsive interactions, when the healing length $\xi := \sqrt{\frac{\hbar^2}{2m\mu}}$ satisfies $\xi \gtrsim \sigma$ and $\xi \ll L$, we introduce the following approximation scheme based on the the effective potential $W(x)$:

$$|\psi_0^{\mathrm{TF,ll}}(x)| \approx \begin{cases} c_{\mathrm{TF,ll}}\sqrt{\frac{\mu-W(x)}{g(N-1)}} & \mu \geq W(x) \\ 0 & \mu < W(x) \end{cases}\,, \qquad (11)$$

where $c_{\mathrm{TF,ll}}$ denotes the normalization constant. In Panels (c)-(f) of Fig. 1, $\psi_0^{\mathrm{TF,ll}}(x)$ is compared against the perturbative approximation, ψ_0^s proposed in Ref. [6]. We further prove that, for given parameters of H^{sp}, there exists a value of $g(N-1)$ at which ψ_0 is well predicted by the normalized localization landscape:

$$\psi_i^{\mathrm{ll}}(x) = \frac{u(x)}{\int_{-L/2}^{L/2}|u(x)|^2\,\mathrm{d}x}\,. \qquad (12)$$

For strong repulsive interactions ($\xi \ll \sigma$), the kinetic energy term in the GPE can be neglected and the wavefunction follows the modulations of the original potential $V(x)$, thus is well described by the Thomas-Fermi approximation [6].

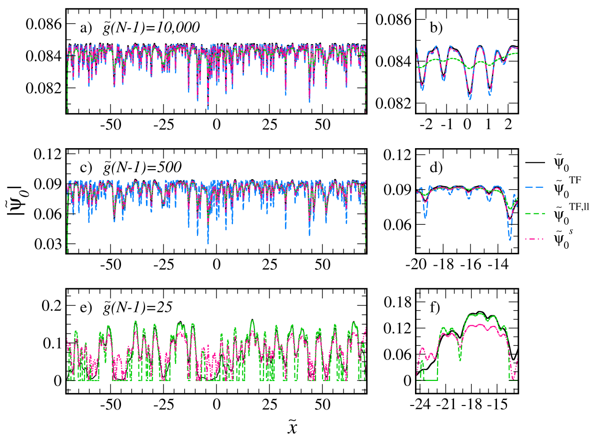

Figure 1: Approximations of $|\tilde{\psi}_0| := |\psi_0| l_\perp^{1/2}$ plotted as functions of $\tilde{x} := x/l_\perp$ for different values of the nonlinear coefficient.

For strong disorder and intermediate interactions the Lifshitz glass phase [7] is achieved, where Bose gas splits into condensates lying in different wells of the effective potential. We numerically show that, in those conditions, a finite number of nonoverlapping SP eigenstates, belonging to the Lifshitz tails of the IDoS, are occupied.

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