Asymptotic models for time-domain scattering by small particles

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Abstract

We propose a new asymptotic model for two-dimensional sound-soft scattering by small circular particles in the time domain. It generalizes existing Foldy-Lax models, enjoys uniform stability properties, and is second-order accurate.

Keywords: sound-soft scattering, asymptotic models, time domain

1 Introduction

We consider the problem of time-domain sound-soft scattering by small circular particles, in the regime when the size of the particles tends to zero. For the moment only very few related works exist in the time domain, see e.g. [1, 3].

2 Problem setting

Let $N \in \mathbb{N}^*$, and $R_j > 0$, $j = 1, \ldots, N$. We denote by $B(\boldsymbol{a}, r) := \{\boldsymbol{x} : \|\boldsymbol{x} - \boldsymbol{a}\| < r\}$, and consider the system of N circles, parametrized by a small parameter $\varepsilon > 0$,

$$\Omega^\varepsilon = \cup_{j=1}^N B(\boldsymbol{c}_j, \varepsilon R_j), \quad \boldsymbol{c}_j \in \mathbb{R}^2,\ j = 1, \ldots N.$$

Let $\Omega^{\varepsilon,c} := \mathbb{R}^2 \setminus \overline{\Omega^\varepsilon}$, and

$$\Gamma_j^\varepsilon = \partial B(\boldsymbol{c}_j, \varepsilon R_j), \quad \Gamma^\varepsilon = \cup_j \Gamma_j^\varepsilon.$$

We look for $u_\varepsilon : \mathbb{R}_+^* \times \Omega^{\varepsilon,c} \to \mathbb{R}$ solving

$$\begin{aligned} &\partial_t^2 u_\varepsilon - \Delta u_\varepsilon = 0, \quad \text{in} \quad \mathbb{R}_+^* \times \Omega^{\varepsilon,c}, \\ &\gamma_0 u_\varepsilon(t) = -\gamma_0 u^{inc}(t), \quad t \in \mathbb{R}_+, \\ &u_\varepsilon(0) = \partial_t u_\varepsilon(0) = 0 \quad \text{in} \quad \Omega^{\varepsilon,c}, \end{aligned} \tag{1}$$

where $u^{inc} : \mathbb{R}_+^* \times \mathbb{R}^2 \to \mathbb{R}$ is a known solution of the homogeneous wave equation in $\mathbb{R}^2$. Evidently, $\lim\limits_{\varepsilon \to 0} \|u_\varepsilon(t)\|_{L^2(\Omega^{\varepsilon,c})} = 0$, and it can be demonstrated that there exist u^{inc} and $t > 0$, s.t. $\|u_\varepsilon(t)\|_{L^2(\Omega^{\varepsilon,c})} \gtrsim \log^{-1} \varepsilon^{-1}$; thus, approximating the field by zero leads to an error of $O(\log^{-1} \varepsilon)$. Our goal is to find a higher-order approximation u_ε^{app} to u_ε.

3 Motivation: one Foldy-Lax model

Since the problem (1) had been rather extensively studied in the frequency domain, it is natural to ask whether time-domain counterparts of the available Foldy-Lax models are of practical interest. We would like to illustrate that such models may be not robust. For this we consider the model [2], which is $O(\varepsilon \log^{-1} \varepsilon)$-accurate. The time-domain counterpart of the Foldy-Lax model of [2] approximates the field u_ε by the superposition of the point sources

$$u_\varepsilon^{FL}(t, \boldsymbol{x}) = \sum_{k=1}^N \mathcal{G}_t(\|\boldsymbol{x} - \boldsymbol{c}_k\|) * \mu_{\varepsilon,k},$$

where $\mathcal{G}_t(d) = \frac{\mathbf{1}_{t>d}}{2\pi\sqrt{t^2 - d^2}}$, and the unknown point densities $\mu_{\varepsilon,k} : \mathbb{R}_+ \to \mathbb{R}$ solve

$$\begin{aligned} -u^{inc}(\boldsymbol{c}_k, t) = &\ \mathcal{G}_t(r_k) * \mu_{\varepsilon,k}(t) \\ &+ \sum_{n \neq k} \mathcal{G}_t(\|\boldsymbol{c}_k - \boldsymbol{c}_n\|) * \mu_{\varepsilon,n}(t). \end{aligned}$$

Unfortunately, this model may exhibit instabilities for some geometric configurations, when the particles are close to each other.

Lemma 1 *Let $N = 3$, and $R_i = 1$, $\forall i$. Let $\varepsilon > 0$ and $\|\boldsymbol{c}_i - \boldsymbol{c}_j\| = c > 0$ for all $i \neq j$ and $c/\varepsilon < 4$. Then there exist $u^{inc} \in C_0^\infty(\mathbb{R}_+, C_0^\infty(\mathbb{R}^2))$ and $\alpha, A > 0$ s.t. $\limsup\limits_{t \to +\infty} \|\mathrm{e}^{-At} u_\varepsilon^{FL}(t)\|_{L^2(\mathbb{R}^2)} \geq \alpha$.*

Let us remark that we observed this phenomenon in the numerical simulations as well. To cope with it, we propose an improved model*.

4 Galerkin Foldy-Lax model

4.1 Main idea and derivation

The solution to (1) can be represented as a single-layer potential of the unknown density λ_ε:

$$u_\varepsilon(t, \boldsymbol{x}) = \int_{\Gamma^\varepsilon} \int_0^t \mathcal{G}_{t-\tau}(\|\boldsymbol{x} - \boldsymbol{y}\|) \lambda_\varepsilon(\tau, \boldsymbol{y}) d\Gamma_{\boldsymbol{y}} d\tau.$$

The continuity of the trace of the single-layer potential yields the the following boundary-integral equation for the density λ_ε: for $\boldsymbol{x} \in \Gamma^\varepsilon$, $t \in \mathbb{R}_+$,

$$-\gamma_0 u^{inc}(t, \boldsymbol{x}) = \int_{\Gamma^\varepsilon} \int_0^t \mathcal{G}_{t-\tau}(\|\boldsymbol{x} - \boldsymbol{y}\|) \lambda_\varepsilon(\tau, \boldsymbol{y}) d\Gamma_{\boldsymbol{y}} d\tau.$$

*I am greatful to Patrick Joly (POEMS) for the suggestion of the name of the new model

Because the functional framework associated to the above problem is somewhat subtle, we rewrite it in the frequency domain. With $\hat{b}(\omega)$ denoting the Fourier-Laplace transform of $b(t)$, i.e. $\hat{b} = \mathcal{F}b$, the above integral equation can be rewritten as: find $\hat{\lambda}_\varepsilon \in H^{-1/2}(\Gamma^\varepsilon)$, s.t.

$$-\gamma_0 \hat{u}^{inc}(\omega, \boldsymbol{x}) = \frac{i}{4} \int_{\Gamma^\varepsilon} H_0^{(1)}(\omega\|\boldsymbol{x}-\boldsymbol{y}\|)\hat{\lambda}_\varepsilon(\omega, \boldsymbol{y}) d\Gamma_{\boldsymbol{y}}.$$

Rewriting the above as: for all $\phi \in H^{-1/2}(\Gamma^\varepsilon)$,

$$-\langle \gamma_0 \hat{u}^{inc}(\omega), \phi \rangle = \langle S(\omega)\hat{\lambda}_\varepsilon, \phi \rangle \tag{2}$$

we remark that the sesquilinear form in the right-hand side is coercive for each $\omega \in \mathbb{C} : \Im\omega > 0$. The respective continuity estimates on $\hat{\lambda}_\varepsilon$ can be made explicit in ω and $\Im\omega$; this allows to translate them into the time domain as stability estimates for λ_ε. The coercivity of (2) translates straightforwardly to its conforming discretizations, and thus automatically yields the stability of the corresponding discretization. Let

$$\mathbb{S}_0(\Gamma_k^\varepsilon) := \{\phi \in H^{-1/2}(\Gamma_k^\varepsilon) : \phi = \text{const}\},$$
$$\mathbb{S}_0^\varepsilon := \prod_{k=1}^{N} \mathbb{S}_0(\Gamma_k^\varepsilon).$$

We discretize (2) on this space: we look for $\hat{\lambda}_\varepsilon^{app} \in \mathbb{S}_0^\varepsilon$, which solves

$$-\langle \gamma_0 \hat{u}^{inc}(\omega), \phi \rangle = \langle S(\omega)\hat{\lambda}_\varepsilon, \phi \rangle, \quad \forall \phi \in \mathbb{S}_0^\varepsilon. \tag{3}$$

The time-domain counterpart of (3) is then called the *Galerkin Foldy-Lax model.* Denoting by $\hat{\lambda}_{\varepsilon,k}^{app} = \hat{\lambda}_\varepsilon^{app}\Big|_{\Gamma_k^\varepsilon}$, we obtain the following convolutional system of equations for unknown functions $\lambda_{\varepsilon,k}^{app}$:

$$-\int_{\Gamma_k^\varepsilon} u^{inc}(t, \boldsymbol{x}) d\Gamma_x = \sum_{n=0}^{N} \int_0^t G_{kn}^\varepsilon(t-\tau)\lambda_{\varepsilon,n}^{app}(\tau) d\tau,$$
$$G_{kn}^\varepsilon(t) = \iint_{\Gamma_k^\varepsilon \times \Gamma_n^\varepsilon} \mathcal{G}_t(\|\boldsymbol{x}-\boldsymbol{y}\|) d\Gamma_{\boldsymbol{y}} d\Gamma_{\boldsymbol{x}}.$$

Knowing the densities $\lambda_{\varepsilon,k}^{app}$ allows to compute u_ε^{app} according to the following rule:

$$u_\varepsilon^{app}(\boldsymbol{x}) = \sum \int_0^t \left(\int_{\Gamma_k^\varepsilon} \mathcal{G}_{t-\tau}(\|\boldsymbol{x}-\boldsymbol{y}\|) d\Gamma_{\boldsymbol{y}} \right) \times \lambda_{\varepsilon,n}^{app}(\tau) d\tau.$$

4.2 Circular particles

While the above procedure works for particles of arbitrary shape, for the circles the Galerkin Foldy-Lax densities solve the equation of a particularly simple form (in the frequency domain):

$$-\int_{\Gamma_k^\varepsilon} \hat{u}^{inc} d\Gamma_{\boldsymbol{x}} = i\pi^2 r_k^2 H_0^{(1)}(\omega r_k) J_0(\omega r_k) \hat{\lambda}_{\varepsilon,k}^{app}$$
$$+ i\pi^2 r_k \sum_{n\neq k} r_n H_0^{(1)}(\omega c_{kn}) J_0(\omega r_n) J_0(\omega r_k) \hat{\lambda}_{\varepsilon,n}^{app}.$$

Surprisingly, with a suitable change of variables, the model of [2] can also be rewritten in the above form, modulo the term $J_0(\omega r_n)$ in the last sum: this term is responsible for the instabilities in the time domain.

4.3 Convergence

We can show the following convergence result.

Theorem 2 *Let K be compact,* $\operatorname{dist}(K, \Omega^{\varepsilon_0,c}) > 0$ *for some $\varepsilon_0 > 0$. As $\varepsilon \to 0$,*

$$\|u_\varepsilon - u_\varepsilon^{app}\|_{L^\infty(0,T;L^\infty(K))} \leq \varepsilon^2 \times C\|u^{inc}\|_{H^9(0,T;H^3(\mathbb{R}^2))},$$

with $C > 0$ depending polynomially on T, N, inverse distance between particles, $\min R_j$, $\max R_j$.

5 Prospectives

We plan to consider the method for more general particle shapes, as well as study the case of closely located particles.

References

[1] H. Barucq, J. Diaz, V. Mattesi, S. Tordeaux, Asymptotic behavior of acoustic waves scattered by very small obstacles, ESAIM Math. Model. Numer. Anal., **55** (2021).

[2] M. Cassier and C. Hazard, Multiple scattering of acoustic waves by small sound-soft obstacles in two dimensions: mathematical justification of the Foldy-Lax model, Wave Motion, **50** (2013), pp. 18–28.

[3] M. Sini, H. Wang, and Q. Yao, Analysis of the acoustic waves reflected by a cluster of small holes in the time-domain and the equivalent mass density, Multiscale Model. Simul., **19** (2021), pp. 1083–1114.