Acoustic passive cloaking using thin resonant ligaments

Lucas Chesnel[1], **Jérémy Heleine**[1,*], **Sergei A. Nazarov**[2]
[1]Inria, Ensta Paris, Institut Polytechnique de Paris, Palaiseau, France
[2]Saint-Petersburg State University, St. Petersburg, Russia
[*]Email: jeremy.heleine@ensta-paris.fr

Abstract

We consider the propagation of acoustic waves in a 2D waveguide unbounded in one direction and containing a bounded obstacle. The goal of this work is to propose a method to cloak the obstacle at infinity, that is to retrieve the reflection and transmission coefficients as in the reference strip.

Keywords: acoustic waveguide, passive cloaking, asymptotic analysis, thin resonators, scattering coefficients, complex resonance

1 Introduction

Let Ω denote a 2D waveguide, unbounded in the (Ox) direction and containing a compact obstacle (rigid or penetrable inclusion, deformation of the wall, ...). In other words, the obstacle is located inside a region $\{(x, y) \in \Omega; |x| < L\}$, for a given $L > 0$, and Ω coincides with the reference guide $\Omega_0 = \mathbb{R} \times (0, 1)$ outside of this region. Considering the propagation of acoustic waves in Ω leads us to study the problem

$$\left\{\begin{array}{rcll} \Delta u + k^2 u &=& 0, & \text{in } \Omega, \\ \partial_{\mathbf{n}} u &=& 0, & \text{on } \partial\Omega, \end{array}\right. \tag{1}$$

where $\mathbf{n}$ denotes the outer unit normal to Ω. We fix the wavenumber $k \in (0, \pi)$ so that only the modes $w^{\pm}\colon (x, y) \mapsto \mathrm{e}^{\pm ikx}$ can propagate. We are interested in the solutions $u^{\pm}$ to (1) generated by the waves $w^{\pm}$ coming from $\mp\infty$. They admit the decompositions

$$u^+ = \left|\begin{array}{ll} w^+ + R^+ w^- + \dots, & x \to -\infty, \\ \quad T w^+ + \dots, & x \to +\infty, \end{array}\right.$$

$$u^- = \left|\begin{array}{ll} \quad T w^- + \dots, & x \to -\infty, \\ w^- + R^- w^+ + \dots, & x \to +\infty, \end{array}\right.$$

where the ellipsis stand for evanescent terms and $R^{\pm}, T \in \mathbb{C}$ are reflection and transmission coefficients.

In the reference guide, the solution is simply $u = w^+$ so that we have zero reflection and a transmission coefficient equal to one. The goal of this work is to explain how to cloak the obstacle by perturbing the geometry.

The difficulty in this task lies in the fact that in general the dependence of the scattering coefficients with respect to the geometry is implicit and not linear. Our approach consists in adding thin ligaments of width $\varepsilon \ll 1$ (see Figure 1). We then create a new geometry Ω^ε where we denote by u^ε the solution to (1) and by R^ε, T^ε its scattering coefficients. We will see that, since ligaments are almost 1D objects, we can get relatively explicit formulas allowing us to find situations where, as $\varepsilon \to 0$:

$$R^\varepsilon = o(1), \quad T^\varepsilon = 1 + o(1),$$

as if, approximately, there were no obstacle.

2 Asymptotic analysis

Attach the ligament $\mathcal{L}^\varepsilon = \left(p - \frac{\varepsilon}{2}, p + \frac{\varepsilon}{2}\right) \times (1, 1 + \ell^\varepsilon)$ at the point $A = (p, 1)$, $p \in \mathbb{R}$. We assume that its length is equal to $\ell^\varepsilon = \ell + \varepsilon \ell'$. Here ℓ, ℓ' will be fixed later to obtain interesting phenomena.

The first step is to derive an asymptotic expansion of u^ε, R^ε and T^ε, as $\varepsilon \to 0$. We employ the technique of matched asymptotic expansions. In the process, the following 1D problem plays a key role:

$$\left\{\begin{array}{rl} v'' + k^2 v = 0, & \text{in } (1, 1 + \ell), \\ v(1) = v'(1 + \ell) = 0. & \end{array}\right. \tag{2}$$

It is obtained by considering the restriction of the Helmholtz problem to the ligament. We choose ℓ as a resonant length of (2), i.e. such that $k\ell = \frac{\pi}{2} + m\pi$, $m \in \mathbb{N}$. In this case, (2) admits non zero solutions.

Theorem 1 *When ε tends to zero, we have :*

$$u^\varepsilon = u^+ + a(\ell') k \gamma + o(1) \text{ in } \Omega,$$

$$u^\varepsilon = \varepsilon^{-1} a(\ell') \sin(k(\cdot - 1)) + O(1) \text{ in } \mathcal{L}^\varepsilon,$$

$$R^\varepsilon = R^+ + i \frac{a(\ell')}{2} u^+(A) + o(1),$$

$$T^\varepsilon = T + i \frac{a(\ell')}{2} u^-(A) + o(1),$$

where $a(\ell') \in \mathbb{C}^*$ and γ is a certain Green function centered at A.

3 Zero reflection

When ℓ^ε varies around the resonant length ($\ell' \in \mathbb{R}$), the main terms in the asymptotics of R^ε, T^ε run on circles whose features depend in an explicit way on the position A of the ligament. We can then find out situations where one can cancel asymptotically the reflection (see Figure 1).

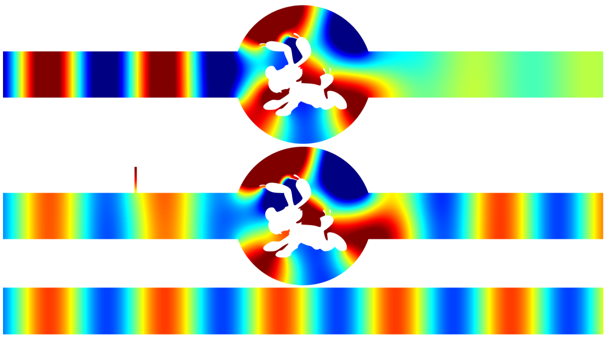

Figure 1: Top: the obstacle generates a rather large reflection. Middle: a thin ligament is added and tuned to get $R^\varepsilon \approx 0$.

Theorem 2 *Assume that $R^+ \neq 0$ and $T \neq 0$. Then there are some positions at which one can place the ligament and tune its length to get $R^\varepsilon = o(1)$.*

The relation of conservation of energy $|R^\varepsilon|^2 + |T^\varepsilon|^2 = 1$ implies that if $R^\varepsilon = o(1)$, we get $|T^\varepsilon| = 1 + o(1)$. As a consequence, in general we only have $T^\varepsilon = e^{i\varphi} + o(1)$ for some $\varphi \neq 0$: there is a phase shift in the transmission (see the right part of the guides in Figure 1).

4 Phase shifter

To compensate this phase shift, let us create a phase shifter. To proceed, we perturb the reference guide $\Omega_0 = \mathbb{R} \times (0, 1)$ by adding two ligaments.

Theorem 3 *Let $\mu \in (-\pi, \pi)$. Then one can place two ligaments and tune their lengths to get*

$$R^\varepsilon = o(1) \quad \text{and} \quad T^\varepsilon = e^{i\mu} + o(1).$$

Now, by combining both Theorem 2 and Theorem 3, one can approximately cloak an obstacle appealing to the following procedure:

1. cancel the reflection with one ligament; this leads to $T^\varepsilon = e^{i\varphi} + o(1)$;
2. far to the right of the obstacle (to neglect the evanescent terms), place two ligaments that do not produce reflection and create a phase shift equal to $-\varphi$.

5 Cloaking with only two ligaments

The previous procedure guarantees that one can cloak an obstacle with three ligaments. Actually, it is also possible to show that cloaking is achievable with only two resonators. For that, the following result is necessary.

Theorem 4 *Assume that T is located on the circle of center $1/2$ and of radius $1/2$. Then, there are some positions at which one can place the ligament and tune its length to get $R^\varepsilon = o(1)$ and $T^\varepsilon = 1 + o(1)$.*

This shows that, for a specific class of obstacles, only one ligament is necessary to achieve cloaking. Now, the following procedure can be used to cloak any obstacle with two ligaments:

1. place a ligament and tune its length to get a transmission coefficient on the circle of center $1/2$ and of radius $1/2$;
2. place another ligament and tune its length to cancel the new reflection.

This procedure has been used in Figure 2. We observe that, both on the left and right of the guide, the wave has the same behavior as in the reference guide. For more details, we refer the reader to [1].

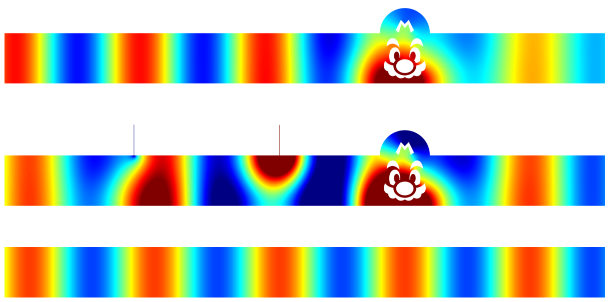

Figure 2: Top: scattering of w^+ in presence of the initial obstacle. Middle: two ligaments have been added and tuned to get $R^\varepsilon \approx 0$ and $T^\varepsilon \approx 1$.

References

[1] L. Chesnel, J. Heleine and S.A. Nazarov, Acoustic cloaking using thin resonant ligaments, submitted, hal-03216053.