The Hermite-Taylor Correction Function Method for Maxwell's Equations

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Abstract

A new approach, based on the Correction Function Method (CFM), to handle boundary conditions for Hermite-Taylor methods is proposed. In Hermite methods not only the solution but also its derivatives need to be determined on the boundary. Here we provide additional information to determine all degrees of freedom on the boundary by a pre-computation where a minimization problem is solved. The functional to be minimized is a square measure of the residual associated with the original problem, that is Maxwell's equations in this work. Numerical examples are performed in 1-D and the expected convergence order is obtained.

Keywords: Hermite methods, Correction function method, Maxwell's equations, High order, Boundary conditions

1 Introduction

Hermite methods achieve arbitrary order of accuracy while maintaining a stability condition only depending on the largest wave-speed, independent of the order of the method [1]. However, the imposition of general boundary conditions is cumbersome (and largely unexplored) since a $(2\,m+1)$-order Hermite method requires the knowledge of all electromagnetic fields and their m first derivatives on the boundary. A possible solution to this is to use a hybrid DG-Hermite method as in [2] but this does require the use of local timestepping to maintain the large timesteps in the Hermite method. Here, we propose an alternative solution based on the CFM [3,4] to handle boundary conditions.

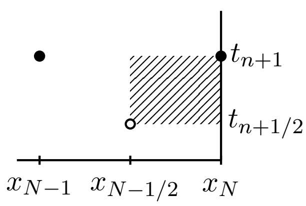

Figure 1: Local CFM patch for the boundary node x_N. The dashed box is $\Omega_h \times I_\Gamma^h$.

Let us assume a domain $\Omega = [x_\ell, x_r]$ and a time interval $I = [t_0, t_f]$, we focus on the following form of Maxwell's equations in 1-D:

$$\begin{aligned}
\mu\,\partial_t H + \partial_x E &= 0 \quad \text{in} \quad \Omega\times I,\\
\epsilon\,\partial_t E + \partial_x H &= 0 \quad \text{in} \quad \Omega\times I,\\
H(x,t_0) &= a(x), \quad \forall x\in\Omega,\\
E(x,t_0) &= b(x), \quad \forall x\in\Omega,\\
c_1\,E(x_\ell,t) + c_2\,H(x_\ell,t) &= g_\ell(t), \quad \forall t>t_0,\\
c_3\,E(x_r,t) + c_4\,H(x_r,t) &= g_r(t), \quad \forall t>t_0,
\end{aligned} \tag{1}$$

where H is the magnetic field, E is the electric field, μ is the magnetic permeability, ϵ is the electric permittivity, c_i for $i=1,\dots,4$ are known coefficients, and $a(x)$, $b(x)$, $g_\ell(t)$ and $g_r(t)$ are known functions.

2 Hermite-Taylor Methods

Hermite methods use a staggered grid in space and time defined by a primal grid $x_i = x_\ell + i\,\Delta x$, $i=0,\dots,N$, and a dual grid $x_{i+1/2} = x_\ell + (i+1/2)\,\Delta x$, $i=0,\dots,N-1$. Here $\Delta x = \frac{x_r - x_\ell}{N}$ and N is the number of cells on the primal grid. The approximate solution on the primal grid is centered at times $t_n = t_0 + n\,\Delta t$ while the approximation on the dual grid is centered at times $t_{n+1/2} = t_0 + (n+1/2)\,\Delta t$.

Assuming that the m first derivatives at the initial time are available, we construct Hermite interpolants of degree $2\,m+1$. Afterward, we evolve the interpolant at the cell center through time using a recursive relation between time and space derivatives that comes from the system of PDEs. For linear hyperbolic problems, this evolution is exact. In other words, knowing all space derivatives of a given polynomial approximation at a given point $(x_i, t_{n+1/2})$ allows us to obtain the exact Taylor expansion in time for this polynomial and its derivatives, defining the update at (x_i, t_{n+1}).

3 Imposition of Boundary Conditions

Consider the problem of finding polynomials of degree m at (x_N, t_{n+1}) in Figure 1 that can be used together with the Hermite data at x_{N-1} to interpolate to $(x_{N-1/2}, t_{n+1})$ and subsequently

$$
\begin{aligned}
J(H,E) =& \frac{1}{2}\int_{I_h}\Big[\ell_h\int_{\Omega_h}(\mu\,\partial_t H+\partial_x E)^2+(\epsilon\partial_t E+\partial_x H)^2\,dx+(c_3E(x_r,t)+c_4\,H(x_r,t)-g_r(t))^2\\
&+(H(x_{N-1/2},t)-H^*(x_{N-1/2},t))^2+(E(x_{N-1/2},t)-E^*(x_{N-1/2},t))^2\Big]\,dt
\end{aligned}
\tag{2}
$$

evolve the interpolant in time. Let H^* and E^* be the approximations to the magnetic and electric fields by the Hermite method on the dual grid and centered in time at $t_{n+1/2}$. We then seek space time polynomials E and H such that the functional (2) is minimized. Precisely we solve the following problem at the boundary

$$
\begin{aligned}
&\text{Find } (H,E)\in V\times V \text{ such that}\\
&(H,E)\in \arg\min_{v,w\in V} J(v,w),
\end{aligned}
\tag{3}
$$

where $V = P^k(\Omega_h\times I_h)$ is the space of polynomials of degree k. Note that the functional is consistent with Maxwell's equations (1).

We choose the degree $k\geq 2m$ to preserve the order of the Hermite-Taylor method and note that once the solution to the minimization problem is found the approximations H and E are known in the local patch, as shown in Fig. 1. Finally note that for linear problems in stationary geometry, as we consider here, the solution to the minimization problem simply entails a small linear solve whose factorization can be computed and stored prior to timestepping.

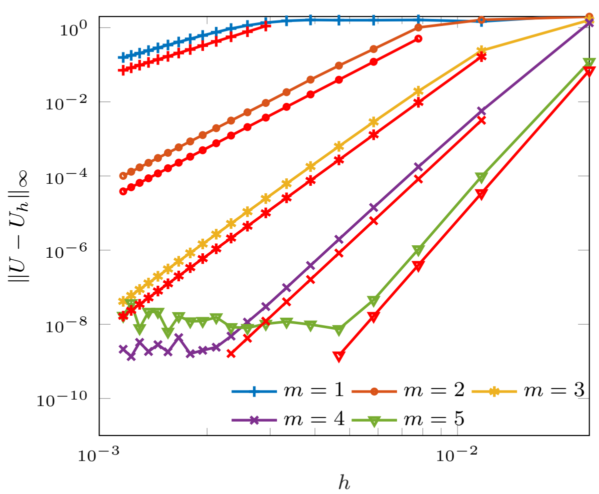

Figure 2: Convergence plots for various values of m. The red curves correspond to the expected order and $U=[H,E]^T$.

4 Numerical Examples

Consider a domain $\Omega=[\frac{1}{3},\frac{8}{3}]$, a time interval $I=[0,\frac{19}{3}]$, $\mu=1$ and $\epsilon=1$. We set $\Delta t=0.5\,h$, $c_1=c_3=0.5$, $c_2=c_4=1$, and set the initial and boundary data so find that the solution to the problem is $H(x,t)=\sin(250x)\sin(250t)$, $E=\cos(250x)\cos(250t)$. Figure 2 shows how the errors follow the expected $2m+1$ rates of convergence.

As a final numerical example, we solve the 2-D Maxwell's equations in a 2×2 PEC cavity with the Hermite-CFM of order of accuracy 3. As can be seeing in Figure 3, the errors are smooth and evenly distributed over the computational domain, indicating that the boundary treatment is stable and accurate.

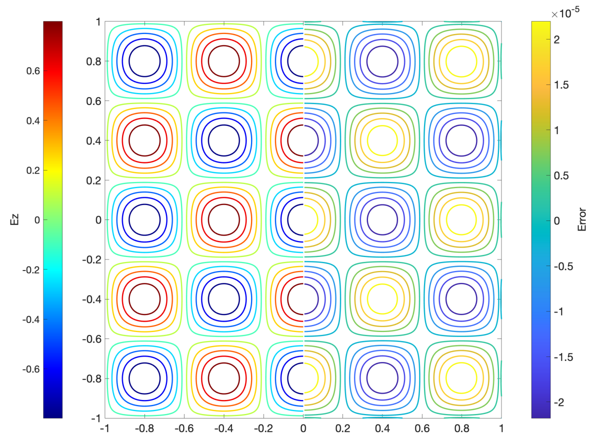

Figure 3: Solution (left) and error (right) for a cavity problem in 2D.

References

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