EM-WaveHoltz: A time-domain frequency-domain solver for Maxwell's equations

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Abstract

The EM-WaveHoltz method for computing time-harmonic solutions of Maxwell's equations by time-domain simulations is presented. Numerical examples illustrating the properties of EM-WaveHoltz are given.

Keywords: Maxwell's equations, frequency-domain, time-domain, positive definite.

1 The WaveHoltz iteration for Maxwell

Maxwell's frequency-domain equations closed by boundary conditions corresponding to either a perfect electric conductor or to an unbounded domain take the form

$$i\omega\epsilon\mathbf{E} = \nabla \times \mathbf{H} - \mathbf{J}, \tag{1a}$$

$$i\omega\mu\mathbf{H} = -\nabla \times \mathbf{E}. \tag{1b}$$

Here $\mathbf{E}$ and $\mathbf{H}$ are the complex valued electric and magnetic fields, ϵ, μ are real valued permittivity and permeability and $\mathbf{J}$ is the real valued current source.

Let $T = 2\pi/\omega$-periodic fields $\widetilde{\mathbf{E}} = \hat{\mathbf{E}}_0 \cos(\omega t) + \hat{\mathbf{E}}_1 \sin(\omega t)$, $\widetilde{\mathbf{H}} = \hat{\mathbf{H}}_0 \cos(\omega t) + \hat{\mathbf{H}}_1 \sin(\omega t)$, be solutions of the time-domain equations

$$\epsilon\partial_t\widetilde{\mathbf{E}} = \nabla \times \widetilde{\mathbf{H}} - \sin(\omega t)\mathbf{J}, \tag{2a}$$

$$\mu\partial_t\widetilde{\mathbf{H}} = -\nabla \times \widetilde{\mathbf{E}}. \tag{2b}$$

Then $\Im\{\mathbf{E}\} = \hat{\mathbf{E}}_0$, $\Im\{\mathbf{H}\} = \hat{\mathbf{H}}_0$, $\Re\{\mathbf{E}\} = \hat{\mathbf{E}}_1 = \frac{1}{\epsilon}\nabla \times \hat{\mathbf{H}}_0$, and $\Re\{\mathbf{H}\} = \hat{\mathbf{H}}_1 = -\frac{1}{\mu}\nabla \times \hat{\mathbf{E}}_0$.

Building on the ideas introduced in [1] our EM-WaveHoltz method finds the periodic solutions by iteratively determining the initial data to (2). Let $\boldsymbol{\nu} = (\boldsymbol{\nu}_E, \boldsymbol{\nu}_H)^T$ be initial conditions to (2). Then the filter operator, Π, is defined

$$\Pi\boldsymbol{\nu} = \Pi\begin{pmatrix}\boldsymbol{\nu}_E\\ \boldsymbol{\nu}_H\end{pmatrix} = \frac{2}{T}\int_0^T \left(\cos(\omega t) - \frac{1}{4}\right)\begin{pmatrix}\widetilde{\mathbf{E}}_{\boldsymbol{\nu}}\\ \widetilde{\mathbf{H}}_{\boldsymbol{\nu}}\end{pmatrix} dt.$$

Here $T = 2\pi/\omega$ and $\widetilde{\mathbf{E}}_{\boldsymbol{\nu}}$ and $\widetilde{\mathbf{H}}_{\boldsymbol{\nu}}$ are the solution to (2) with initial conditions $\boldsymbol{\nu} = (\boldsymbol{\nu}_E, \boldsymbol{\nu}_H)^T$. The operator Π is contractive and can be used to define the EM-WaveHoltz iteration

$$\boldsymbol{\nu}^{n+1} = \Pi\boldsymbol{\nu}^n, \quad \text{with} \quad \boldsymbol{\nu}^0 = (\boldsymbol{\nu}_E^0, \boldsymbol{\nu}_H^0)^T = \mathbf{0}. \tag{3}$$

The EM-WaveHoltz iteration converges to the imaginary parts of the solution to the frequency-domain equation

$$\lim_{n\to\infty} \boldsymbol{\nu}^n = \lim_{n\to\infty} (\boldsymbol{\nu}_E^n, \boldsymbol{\nu}_H^n)^T = (\Im\{\mathbf{E}\}, \Im\{\mathbf{H}\})^T,$$

and the real parts can be recovered via the expressions above.

It is easy to rewrite the fixed point problem as a positive definite linear system that can be efficiently solved by a Krylov subspace method. To see this define $S\boldsymbol{\nu} \equiv \Pi\boldsymbol{\nu} - \Pi\mathbf{0}$. We can then write $\Pi\boldsymbol{\nu} = S\boldsymbol{\nu} + \Pi\mathbf{0}$, thus finding the fix point of Π: $\Pi\boldsymbol{\nu} = \boldsymbol{\nu}$ is equivalent to solving the equation $(I - S)\boldsymbol{\nu} = \Pi\mathbf{0}$. To obtain the right hand side $\Pi\mathbf{0}$, we first solve, (using your favourite Maxwell solver) the time-domain problem (2) with zero initial conditions $\boldsymbol{\nu} = \mathbf{0}$ from $t = 0$ to $t = T = 2\pi/\omega$ once. The filter is computed using the trapezoidal rule. Similarly the cost to compute one Krylov vector is that of a wave solve with initial data. In our method we can choose to make the Krylov subspace smaller by noting that although we are looking for a $T = \frac{2\pi}{\omega}$-periodic solution, there is nothing in the method that prevents us from changing the filtering to extend over a longer time, say, $T = N_{\text{periods}}\frac{2\pi}{\omega}$. As we show in the numerical examples below, for moderate N_{periods} this reduces the number of iterations by a factor of roughly N_{periods} so that the overall computational cost is almost the same, but the memory consumption is N_{periods} times smaller.

2 Numerical Examples

2.1 Comparison with MEEP

We fist compare our method with the iterative Yee-FDFD solver of the open source C++ package MEEP [2]. Our code is implemented by combining EM-WaveHoltz with the Yee scheme of the C library RBCpack [3]. Our code uses GMRES without restart. The FDFD solver of MEEP uses the BICG-Stab(l) method.

Following MEEP package's benchmark example for the FDFD code we consider a ring

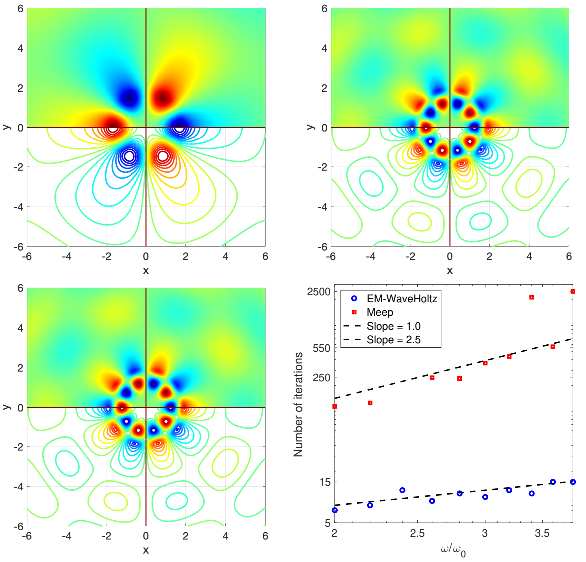

Figure 1: The real part of the E_z field (normalized) for $\omega = \omega_0, 2.24\omega_0, 2.7\omega_0$, Left part of the figures is Yee-EM-WaveHoltz, and MEEP's FDFD solver is to the right.

resonator and the 2D TM model. The computational domain is $[-6,6]^2$ with nonreflecting boundary conditions, [3]. A ring resonator with $\epsilon_r = 3.4^2$ is located at $\{(x,y) : 1 \le \sqrt{x^2+y^2} \le 2\}$. The permittivity outside the ring is $\epsilon = 1$, and the permeability $\mu = 1$ in the whole computational domain. Two point sources are placed at $(1.1, 0)$ with magnitude 1 and $(-1.1, 0)$ with magnitude -1. We consider $\omega = \omega_0, 2.24\omega_0$ and

Table 1: Computational time (sec)

	N	$\omega = \omega_0$	$2.24\omega_0$	$2.7\omega_0$
EM-WH	120	20	12	12
	240	94	55	60
	480	562	326	350
MEEP	120	11	21	34.14
	240	109	180	229
	480	1095	1531	2000

$2.7\omega_0$, with $\omega_0 = 0.118 \times 2\pi$. The relative tolerance is 10^{-7}. We use $l = 10$ in the MEEP BICG-Stab-(l) FDFD solver. In the results displayed in Figure 1, we observe that the EM-WaveHoltz and MEEP solutions agree well. Table 1 presents the computational time needed (N is the number of gridpoints in each dimension). The Yee-EM-WaveHoltz method is almost always faster and its advantage increases as the solution is more accurate or when the frequency is increased. We sweep $\omega \in [2\omega_0, 3.8\omega_0]$ with step size 0.2. The number of grid points per wavelength is fixed. We use $l = 20$ in the BICS-Stab-(l) and a relative tolerance 10^{-5}. The bottom right figure of Figure 1 shows the scaling of the two solvers.

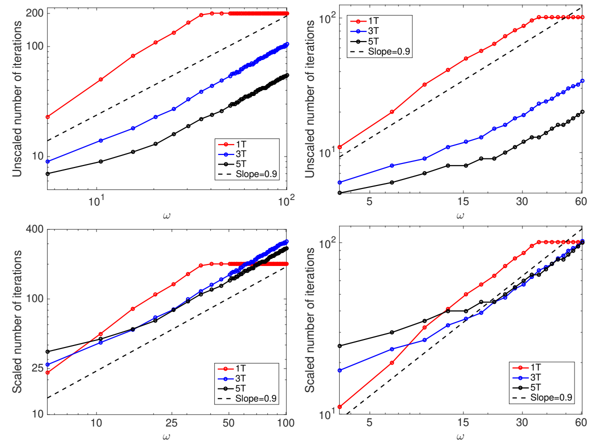

Figure 2: Unscaled (top) and scaled (bottom) number of iterations as a function of frequency for different filtering time. Left: 2D open problem. Right: 3D open problem.

2.2 Smaller Krylov Subspaces

As mentioned, we can filter over multiple periods. We consider $T = N_\text{p}\frac{2\pi}{\omega}$ with $N_\text{p} = 1, 3, 5$ for 2D and 3D open domain problems (same EM-WaveHoltz solver as above). We scan over different frequencies and apply the GMRES accelerated Yee-EM-WaveHoltz. The total number of iteration allowed is set as 200 in 2D and 100 in 3D. In Figure 2, we present the number of iterations as a function of the frequency. We also scale the number of iterations by N_p and present the result in the same Figure. Note that for $N_\text{p} = 3$ and $N_\text{p} = 5$ the scaled curves visually collapse, implying that the the total computational time is approximately the same but with 3 and 5 times smaller memory footprint.

References

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