# Accounting for viscothermal boundary losses in time-domain acoustics

**Linus Hägg**[1,*], **Martin Berggren**[1]
[1]Department of Computing Science, Umeå University, Umeå, Sweden
*Email: linush@cs.umu.se

## Abstract

It has recently been shown that viscothermal boundary losses may be accurately and efficiently captured by altering the boundary condition at solid walls in standard frequency-domain simulations of the acoustic pressure Helmholtz equation. Here, we investigate the corresponding time-domain boundary condition involving fractional differintegral operators. We analyze an initial-boundary-value problem with thermal boundary losses and report numerical experiments.

**Keywords:** acoustics, viscothermal boundary losses, Riemann–Liouville fractional integral

## 1 Introduction

Viscothermal boundary losses may have a significant impact on sound propagation in narrow geometries. In these situations sound propagation may be modeled by the linearized compressible Navier–Stokes equations. Unfortunately, it is computationally expensive to obtain numerical solutions to the linearized Navier–Stokes equations due to the high resolution needed to resolve the thermal and viscous boundary layers. Starting with the work of Cremer [2], simplified models that are appropriate close to the walls of the domain have been derived based on acoustic boundary layer theory. A notable contribution in this respect is the approximate boundary condition derived by Pierce [6] for single-frequency sound propagation. Recently, Berggren et al. [1] proposed a computationally effective procedure by rederiving and implementing Pierce's boundary condition to account for viscothermal boundary losses in standard quiescent, linear frequency-domain acoustics, that is, as a boundary condition for the pressure Helmholtz equation.

The frequency-domain boundary condition (phase convention $\exp(i\omega t)$, $\omega > 0$) may be expressed in the (complex) acoustic pressure $\hat{p}$ and velocity $\hat{u}$, satisfying $i\omega\rho_0\hat{u} = -\nabla\hat{p}$, as

$$n \cdot \hat{u} = -\frac{c\tau_V}{\sqrt{i\omega\tau_V}}\nabla_\Gamma \cdot \hat{u}_\Gamma + \sqrt{i\omega\tau_T}\frac{\hat{p}}{\rho_0 c}, \qquad (1)$$

where $n$ denotes the exterior unit normal, $\sqrt{i} = (1+i)/\sqrt{2}$, $\nabla_\Gamma \cdot \hat{u}_\Gamma$ the tangential divergence, $\rho_0$ the ambient mass density, $c$ the speed of sound, and $\tau_V$ and $\tau_T$ the viscous and thermal timescales

$$\tau_V = \frac{\nu}{c^2} \text{ and } \tau_T = \frac{(\gamma-1)^2\kappa}{\rho_0 c^2\, c_p}, \qquad (2)$$

where $\nu$ denotes the kinematic viscosity, $\kappa$ the thermal conductivity, $\gamma$ the heat capacity ratio, and $c_p$ the specific heat capacity at constant pressure. The boundary condition may be interpreted as a generalized impedance boundary condition, and the appearance of $\sqrt{i\omega}$ is characteristic of diffusion processes [5]. When appropriate, boundary condition (1) is a small perturbation to the regular wall condition $n \cdot \hat{u} = 0$; in air $\tau_V \sim 10^{-10}$ s and $\tau_T \sim 10^{-11}$ s at atmospheric conditions. Here, we investigate the corresponding boundary condition in time domain.

There is a plethora of versatile time-domain impedance boundary conditions for acoustic simulations; however, these typically require parameter tuning [5]. Moreover, specialized models, such as the Webster–Lokshin equation [3], have been developed for lossy sound propagation in special geometries. However, the boundary condition introduced here is applicable to general geometries and requires no parameter tuning.

## 2 Viscothermal BC in time domain

It is possible to derive the time-domain analogue of boundary condition (1) from the linearized Navier–Stokes equations by repeating, in time domain, the frequency-domain procedure outlined by Berggren et al. [1]. However, the same expression may also be obtained using Fourier transforms [7, expressions (7.1) and (7.4)],

$$\begin{aligned} n \cdot u = &- c\,\nabla_\Gamma \cdot \left(\sqrt{\tau_V}\,{}_{-\infty}D_t^{-1/2} u_\Gamma\right) \\ &+ \sqrt{\tau_T}\,{}_{-\infty}D_t^{1/2}\frac{p}{\rho_0 c}, \end{aligned} \qquad (3)$$

where ${}_{-\infty}D_t^{-1/2}$ and ${}_{-\infty}D_t^{1/2} = \partial_t\,{}_{-\infty}D_t^{-1/2}$ denote the causal Riemann–Liouville fractional integral and derivative of order 1/2 starting at $-\infty$. Typically, specialized discretizations of the nonlocal differintegrals are required to achieve sufficient computational efficiency [4].

## 3 Energy balance including thermal losses

To derive an energy balance that accounts for thermal boundary layer effects using boundary condition (3), we consider an initial–boundary-value problem for the acoustic pressure and scaled velocity $u_p = \rho_0 c\, u$,

$$\partial_t p + \nabla \cdot (c u_p) = 0,\ Q = (0,T) \times \Omega, \quad (4a)$$
$$\partial_t u_p + c\nabla p = 0,\ Q = (0,T) \times \Omega, \quad (4b)$$
$$p = 0, u_p = 0,\ Q_0 = \{0\} \times \Omega, \quad (4c)$$
$$n \cdot u_p - \sqrt{\tau_T}\,{}_0D_t^{1/2} p = 0,\ \Sigma_w = (0,T) \times \Gamma_w, \quad (4d)$$
$$p - n \cdot u_p - 2g = 0,\ \Sigma_{io} = (0,T) \times \Gamma_{io}, \quad (4e)$$

where $T > 0$ is an arbitrary end time, and $g$ is a finite duration source acting at the in/out-boundary part $\Gamma_{io}$, which is complementary to the solid wall $\Gamma_w$. We have assumed that $p \equiv 0$ for $t \leq 0$, so that ${}_{-\infty}D_t^{1/2} p = {}_0D_t^{1/2} p$ in boundary condition (4d). Assuming that $p$ and $u_p$ are sufficiently regular, applying $\int_\Omega p$ to equation (4a), $\int_\Omega u_p \cdot$ to equation (4b), summing, integrating by parts either of the spatial derivatives, rearranging the terms, and invoking the boundary conditions, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(p^2 + |u_p|^2\right) = -\int_{\partial\Omega} cp\, n \cdot u_p$$
$$= \int_{\Gamma_{io}} cg^2 - \int_{\Gamma_{io}} c(p-g)^2 - \int_{\Gamma_w} cp\,\sqrt{\tau_T}\,{}_0D_t^{1/2} p. \quad (5)$$

Multiplying equation (5) by $1/(\rho_0 c^2)$, we find that the rate of change of the acoustic energy is determined by the net influx power, represented by the first two terms in the right side, and the exchange of energy in the thermal boundary layer, represented by the last term. Integrating balance (5) in time, we obtain the energy estimate

$$\int_\Omega (p^2 + |u_p|^2) \leq 2 \int_{\Sigma_{io}} cg^2, \quad (6)$$

provided that the term $\int_{\Sigma_w} cp\,\sqrt{\tau_T}\,{}_0D_t^{1/2} p \geq 0$, that is, the term represents a *thermal boundary loss.* Indeed, we may prove the required positivity using a diffusive representation of the half-derivative [3].

In case the viscous contribution to boundary condition (3) is included, we have not succeeded to derive an energy estimate. In fact, if viscous effects are included, we may demonstrate that an infinite duct subject to boundary condition (3) supports arbitrarily fast growing modes, which indicates ill-posedness of the formulation.

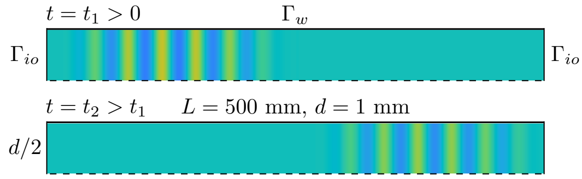


Figure 1: Two snapshots of a wave packet traveling from left to right in a straight duct with thermal boundary losses. The duct has been simulated in planar symmetry; the upper boundary is a solid wall and the lower a symmetry line.

## 4 Numerical experiments

Figure 1 displays two snapshots of a wave packet traveling from left to right in a straight duct with thermal boundary losses, which have been obtained by a FDTD discretization of initial–boundary-value problem (4). As suggested by, for instance, Monteghetti et al. [4], we apply a time-local discretization of the half-derivative based on a diffusive representation, which significantly reduces the computational cost compared to naive discretizations of boundary condition (4d). Transmission characteristics have been successfully verified against frequency-domain simulations using boundary condition (1).

## References


[1] M. Berggren, A. Bernland, & D. Noreland, *J. Comput. Phys.* **371** (2018), 633–650.

[2] L. Cremer, *Arch. Elek. Übertr.* **2** (1948), 136–139.

[3] H. Haddar, & D. Matignon, Rapport de Recherche INRIA 6558 (2008).

[4] F. Monteghetti, D. Matignon, & E. Piot, Appl. Numer. Math **155** (2020), 73–92.

[5] F. Monteghetti, D. Matignon, E. Piot, & L. Pascal, *JASA* **140(3)** (2016), 1663–1674.

[6] A. Pierce, *Acoustics: An Introduction to Its Physical Principles and Applications*, Acoustical Society of America, Melville, NY, 1989.

[7] S. G. Samko, A. A. Kilbas, & O. I. Marichev, *Fractional Integrals and Derivatives*, Gordon Breach, Amsterdam, 1993.