Robust boundary integral equations of Helmholtz decomposition formulations of elastic scattering problems

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Abstract

We present a robust boundary integral equation formulation for elastodynamic scattering problems in two dimensions formulated via Helmholtz decompositions. The main advantage of this formulation is its reliance on Helmholtz layer potentials only, which are simpler than their counterparts that correspond to the fundamental solution of Navier equations.

Keywords: Navier equations, combined field integral equations

1 Introduction

Numerical solutions of elastic scattering problems based on boundary integral equation formulations are attractive alternatives over their volumetric counterparts [4]. Following ideas introduced in [2], we introduce in this paper robust and well-conditioned boundary integral equation formulations of elastic scattering problems that are based on the Helmholtz decomposition of the elastic fields in two dimensions, which allows use to deal only with Helmholtz layer potentials.

2 Helmholtz decomposition formulation of Navier scattering problems

Considering a bounded domain Ω in $\mathbb{R}^2$ whose boundary Γ is a closed Lipschitz curve, we are interested in solving the impenetrable elastic scattering problem in the exterior of Ω, that is look for solutions of the time-harmonic Navier equation

$$\nabla\cdot\boldsymbol{\sigma}(\mathbf{u})+\omega^2\mathbf{u}=0\quad\text{in }\Omega^+:=\mathbb{R}^2\setminus\Omega \tag{1}$$

that satisfy the Kupradze radiation condition at infinity, where $\boldsymbol{\sigma}$ is the stress tensor associated with the field $\mathbf{u}$ and the Lamé constants λ and μ. We assume that on the boundary Γ the solution $\mathbf{u}$ of (1) satisfies the Dirichlet boundary condition

$$\mathbf{u}=-\mathbf{u}^{\rm inc}\quad\text{on }\Gamma.$$

The scattered field can be expressed in the form of the Helmholtz decomposition $\mathbf{u}=\mathbf{u}_p+\mathbf{u}_s$.

$$\mathbf{u}_p:=-\frac{1}{k_p^2}\nabla\,\nabla\cdot\mathbf{u}\qquad \mathbf{u}_s:=\frac{1}{k_s^2}\overrightarrow{\rm curl}\,{\rm curl}\,\mathbf{u} \tag{2}$$

where ${\rm curl}\,\mathbf{u}:=\partial_1u_2-\partial_2u_1$ and $\overrightarrow{\rm curl}\,\varphi=[\partial_2\varphi-\partial_1\varphi]^\top$. Hence, following the ideas in [2] we can look for the fields $\mathbf{u}$ in the form

$$\mathbf{u}=\nabla\varphi_p+\overrightarrow{\rm curl}\,\varphi_s \tag{3}$$

where the scalar functions φ_p and φ_s are radiative solutions of scalar Helmholtz equations in Ω^+ with wavenumbers k_p and k_s respectively. It is straightforward to see that φ_p and φ_s satisfy the following coupled boundary conditions

$$\begin{aligned}\partial_n\varphi_p+\partial_s\varphi_s &= -\mathbf{u}^{\rm inc}\cdot\boldsymbol{n} &&\text{on }\Gamma\\ -\partial_s\varphi_p+\partial_n\varphi_s &= \mathbf{u}^{\rm inc}\cdot\boldsymbol{t} &&\text{on }\Gamma\end{aligned} \tag{4}$$

where ∂_n and ∂_s denote the normal and respectively the tangential derivatives on Γ, whereas $\boldsymbol{n}$ and $\boldsymbol{t}$ denote the unit exterior normal and respectively the unit tangent on Γ. We look for φ_p and φ_s in the form of regularized combined field Helmholtz potentials with wavenumbers k_p and respectively k_s

$$\begin{aligned}\varphi_p &:= DL_{\Gamma,k_p}[Y_pg_p]-SL_{\Gamma,k_p}[g_p]\\ \varphi_s &:= DL_{\Gamma,k_s}[Y_sg_s]-SL_{\Gamma,k_s}[g_s]\qquad\text{in }\Omega^+,\end{aligned}$$

where g_p and g_s are unknown functional densities defined on Γ, and Y_p and Y_s are operators to be specified in what follows. We are led to the following system of BIE for the boundary densities g_p and g_s

$$\begin{aligned}\mathcal{A}_{DH}\begin{bmatrix}g_p\\ g_s\end{bmatrix} &= -\begin{bmatrix}\mathbf{u}^{\rm inc}\cdot\boldsymbol{n}\\ -\mathbf{u}^{\rm inc}\cdot\boldsymbol{t}\end{bmatrix}\\ \mathcal{A}_{DH}^{11} &:= \tfrac{1}{2}I+N_pY_p-K_p^\top\\ \mathcal{A}_{DH}^{12} &:= \tfrac{1}{2}\partial_sY_s-\partial_sV_s-k_s^2\boldsymbol{t}\cdot V_s[\boldsymbol{n}Y_s]-K_s^\top\partial_sY_s\\ \mathcal{A}_{DH}^{21} &:= -\tfrac{1}{2}\partial_sY_p+\partial_sV_p+k_p^2\boldsymbol{t}\cdot V_p[\boldsymbol{n}Y_p]+K_p^\top\partial_sY_p\\ \mathcal{A}_{DH}^{22} &:= \tfrac{1}{2}I+N_sY_s-K_s^\top.\end{aligned} \tag{5}$$

ω	N	Helmholtz decomposition BIE (5)	
		ε_∞ Kite	ε_∞ Starfish
16	64	6.4×10^{-3}	3.2×10^{-3}
16	128	7.4×10^{-7}	2.2×10^{-7}
16	256	8.5×10^{-16}	5.6×10^{-16}
32	128	2.3×10^{-3}	3.5×10^{-3}
32	256	5.2×10^{-9}	8.6×10^{-9}
32	512	4.5×10^{-15}	1.1×10^{-14}

Table 1: Errors in the method of manufactured solution using the Helmholtz decomposition BIE for the smooth kite and starfish geometries using the K-M Nyström discretization with for different values of the frequency ω and material parameter values $\lambda = 1$, $\mu = 1$, at various levels of discretization.

ω	N	Teardrop	
		K-M $p = 3$	QBX
16	128	8.6×10^{-4}	7.3×10^{-3}
16	256	1.0×10^{-4}	1.4×10^{-4}
16	512	1.2×10^{-5}	2.5×10^{-5}
16	1024	1.5×10^{-6}	1.3×10^{-6}
32	256	2.9×10^{-3}	1.7×10^{-3}
32	512	4.7×10^{-4}	3.8×10^{-4}
32	1024	6.0×10^{-5}	3.5×10^{-5}
32	2048	7.6×10^{-6}	3.7×10^{-7}

Table 2: Far-field errors in the method of manufactured solution using the Helmholtz decomposition BIE for the singular teardrop geometry with for different values of the frequency ω and material parameter values $\lambda = 1$, $\mu = 1$, at various levels of discretization.

In equation (5) the subscripts p and s refer to the wavenumbers k_p and respectively k_s in the definition of the corresponding boundary integral operators, V is the Helmholtz single layer BIO, $K^\top$ is the adjoint of the double layer BIO, and N is the hypersingular operator. We establish the following

Theorem 1 *Choosing $Y_p = -2V_{k_p+i\varepsilon_p}$ and $Y_s = -2V_{k_s+i\varepsilon_s}$ with $0 < \varepsilon_p$ and $0 < \varepsilon_s$, the operators $\mathcal{A}_{DH}$ are invertible in $L^2(\Gamma) \times L^2(\Gamma)$ for all frequencies $\omega > 0$ when Γ is a smooth closed curve.*

3 Numerical results

The Nyström discretization of the BIE (5) is rather straightforward, and we present two such strategies, one based on the classical Kussmaul-Martensen (K-M) kernel singularity splitting [3], the other on QBX [1]. In the case of piece-smooth boundaries, we use sigmoid transforms (with polynomial degree p) in conjunction with K-M methods, and Chebyshev meshes together with Clenshaw-Curtis quadratures in connection with QBX Nyström discretizations. We present in Tables 1-2 far field errors achieved by the Nyström discretizations of the Helmholtz decomposition BIE (5) in the context of the method of manufactured solutions. Similar accuracy levels are observed in the case of plane wave incident fields. Furthermore, the BIE formulations (5) behave like integral equations of the second kind, and their numbers of GMRES iterations grow only logarithmically with the frequency in the high frequency regime.

References

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