A reconstruction method for the inverse gravimetric problem

GERBER-ROTH Anthony[1,*], MUNNIER Alexandre[1], RAMDANI Karim[1]
[1]Université de Lorraine, CNRS, Inria, IECL, F-54000 Nancy, France
*Email: anthony.gerber-roth@univ-lorraine.fr

Abstract

We consider an inverse source problem for the 2D Laplace operator, for sources of the form $\mathbb{1}_\omega$. The reconstruction method is based on the computation of the harmonic moments of ω. Although not directly related to wave propagation, this problem can be used to handle by perturbation techniques source inverse problems for Helmholtz equation at low frequencies, and can possibly be adapted to tackle wave problems by replacing harmonic functions by plane waves.

Keywords: Inverse gravimetric problem, Quadrature domains, Harmonic moments.

1 Problem setting

Let ω be an open set included in the unit disk and Γ be the unit circle. Denote by U_ω the solution of

$$-\Delta U_\omega = \mathbb{1}_\omega \quad \text{ on } \mathbb{R}^2,$$

such that $U_\omega(x) = O(\ln|x|)$ at infinity. The function U_ω can be thought of as the gravitational field generated by a uniform mass distribution in ω and is given by the Newtonian potential $U_\omega(x) = (G * \mathbb{1}_\omega)(x) = \int_\omega G(x-y)\,\mathrm{d}y$, where $G(x) := -1/2\pi \ln|x|$ is the fundamental solution of the Laplacian in $\mathbb{R}^2$. The inverse problem we are interested in consists in reconstructing ω from the knowledge of ∇U_ω on Γ. The problem is known to be ill-posed, as uniqueness is not guaranteed in general. However, under either of the two following properties:

(S) ω is star-shaped with respect to its center of gravity.

(C) ω is convex in x_1, i.e the intersection of any straight line parallel to the x_1-axis with ω is an interval.

uniqueness and stability results can be proved (see [4]). More precisely, we have:

Theorem 1 *Assume that ω_1, ω_2 are two domains both satisfying either* (**S**) *or* (**C**). *If $\nabla U_{\omega_1} = \nabla U_{\omega_2}$ on Γ, then $\omega_1 = \omega_2$.*

In addition, if the domains ω_1, ω_2 satisfy property **(S)** and admit a representation of the form $\omega_i = \{r < d_i(\sigma)\}$ (in polar coordinates associated to their centers of gravity), then the following logarithmic-type stability estimate holds:

$$\|d_1 - d_2\|_{L^\infty} \leqslant \kappa |\ln \varepsilon|^{-1/\kappa} \quad (\kappa > 0), \tag{1}$$

provided $\|\nabla U_{\omega_1} - \nabla U_{\omega_2}\|_{L^\infty(\Gamma)} \leqslant \varepsilon$.

2 Reconstruction method

From now on, we assume that ω satisfies property **(S)**. We shall reconstruct ω by generating a sequence of domains $(\omega_N)_N$ satisfying **(S)** and such that $\|\nabla(U_\omega - U_{\omega_N})\|_{L^2(\Gamma)}$ tends to 0.

We first observe that Green's formula leads to the equality:

$$\int_\Gamma \partial_n v\, U_\omega - \partial_n U_\omega\, v = \int_\omega v, \tag{2}$$

for all harmonic functions v on the unit disk. Since the knowledge of ∇U_ω on Γ allows deducing the values of U_ω on Γ (recall that U_ω is harmonic outside Γ), our inverse problem can be rephrased as a shape-from-moments problem: *How to reconstruct a domain ω from the knowledge of its harmonic moments $\int_\omega z^\ell$?* This can be done by choosing a positive integer N and:

(i) computing the weights $c_1, ..., c_N \in \mathbb{C}$ and the nodes $z_1, ..., z_N \in \mathbb{C}$ such that:

$$\forall\, 0 \leqslant \ell \leqslant 2N-1, \qquad \int_\omega z^\ell = \sum_{k=1}^N c_k z_k^\ell.$$

(ii) constructing a domain ω_N which actually satisfies these identities for all $\ell \geqslant 0$ (such a domain does exist and is called a harmonic quadrature domain, as described in the next Section).

As a result we have therefore :

$$\forall\, 0 \leqslant \ell \leqslant 2N-1, \qquad \int_\omega z^\ell = \int_{\omega_N} z^\ell,$$

from which we can prove that $\|\nabla(U_\omega - U_{\omega_N})\|_{L^2(\Gamma)} \to 0$ as $N \to +\infty$.

3 Quadrature domains

A bounded domain Ω in the complex plane is called a harmonic quadrature domain if there exists a finite number of weights $(c_k)_{1\leqslant k\leqslant N}$ and nodes $(x_k)_{1\leqslant k\leqslant N}$ in Ω such that for all harmonic integrable function v:

$$\int_\Omega v = \sum_{k=1}^{N} c_k v(x_k).$$

We refer the interested reader to [3] for details on this notion. Unfortunately, in general, a quadrature domain is not uniquely determined by solely the quadrature nodes x_k and weights c_k. An important subclass of harmonic quadrature domains, for which this uniqueness is ensured, is that of subharmonic quadrature domains. This class is defined as the domains satisfying $\int_\Omega v \geqslant \sum_{k=1}^{N} c_k v(x_k)$ for all integrable subharmonic functions v in Ω, with $c_k \geqslant 0$. In addition, for a subharmonic quadrature domain Ω, the function $U_\Omega = G * \mathbb{1}_\Omega$ satisfies the following properties:

$$-\Delta U_\Omega \leqslant 1 \quad \text{in } \mathbb{R}^2 \tag{3a}$$
$$U_\Omega = U_N \quad \text{in } \mathbb{R}^2 \setminus \Omega \tag{3b}$$
$$U_\Omega \leqslant U_N \quad \text{in } \mathbb{R}^2, \tag{3c}$$

where $U_N = \sum_{k=1}^{N} c_k G(\cdot - x_k)$. Notice that we can recover Ω from U_Ω since $-\Delta U_\Omega = \mathbb{1}_\Omega$.

4 Reconstruction algorithm

We fix some integer $N \geqslant 1$ and:

1. We compute $\int_\omega z^\ell$, for $0 \leqslant \ell \leqslant 2N-1$.
2. We compute $c_k \in \mathbb{R}^+$ and $z_k \in \mathbb{C}$ such that
$$\forall 0 \leqslant \ell \leqslant 2N-1, \quad \sum_{k=1}^{N} c_k z_k^\ell = \int_\omega z^\ell. \tag{4}$$
3. We determine ω_N, the unique subharmonic quadrature domain Ω solving system (3).

The harmonic moments $\int_\omega z^\ell$ are computed from the boundary data using (2). System (4) is solved by Prony's method (see [2]), while the differential inequality (3) is solved using FEM (see [1]). We can prove the following convergence results.

Theorem 2 *If ω is a subharmonic quadrature domain, there exists $N \geqslant 1$ such that $\omega = \omega_N$. If ω is a domain satisfying* (**S**)*, and if the domains ω_N also satisfies property* (**S**)*, then the sequence $(\omega_N)_N$ converges to ω in the sense of* (1).

5 Numerical results

We compute $\mathbb{1}_\Omega = -\Delta U_\Omega$ using FreeFEM++. Figure 1 shows the results for exact (left) and noisy (right) moments with 3 % noise.

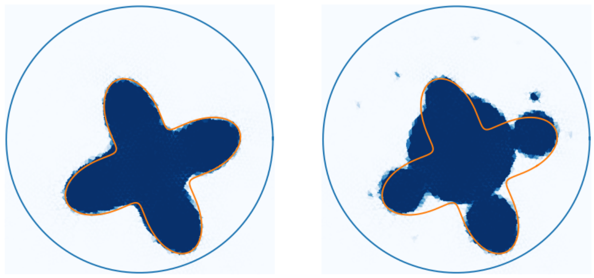

Figure 1: Star-shaped domain with $N = 16$.

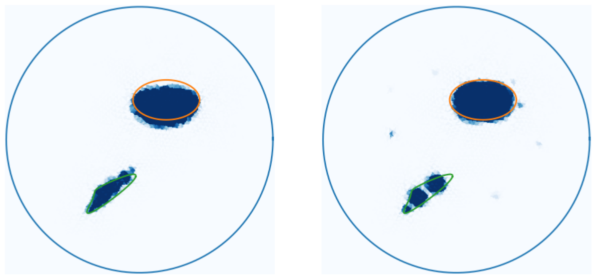

Figure 2: Domain satisfying (**C**) with $N = 18$.

References

[1] K. Atkinson and W. Han. *Theoretical numerical analysis*, volume 39 of *Texts in Applied Mathematics*. Springer, New York, second edition, 2005. A functional analysis framework.

[2] G. H. Golub, P. Milanfar, and J. Varah. A stable numerical method for inverting shape from moments. *SIAM J. Sci. Comput.*, 21(4):1222–1243 (electronic), 1999.

[3] B. Gustafsson and H. S. Shapiro. What is a quadrature domain? In *Quadrature domains and their applications*, volume 156 of *Oper. Theory Adv. Appl.*, pages 1–25. Birkhäuser, Basel, 2005.

[4] V. Isakov. *Inverse problems for partial differential equations*, volume 127. New York, NY: Springer, 2006.