Complex Effective Wavenumbers of Isotropic Particulate Materials

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Abstract

A key assumption used to describe scalar waves in a random particulate material is that the average field satisfies a wave equation with a unique effective wavenumber k_*. By average wave we mean to ensemble average over all possible configurations of particles. As the medium is homogeneous and isotropic - because the particles have no specific orientation and are evenly distributed - it seems reasonable to assume that there is only one effective wavenumber. However, recently two different theoretical models have predicted that there exist *at least two* (complex) effective wavenumbers for one fixed frequency [1], [3]. A phenomenon normally observed only in anisotropic media. Our goal is to find clear evidence of these complex effective wavenumbers by using a Monte Carlo approach based on high fidelity simulations. To achieve this, we place a mix of random particles inside a very large plate geometry. In conclusion, we find evidence for two effective wavenumbers and even discover an effective wavenumber with a negative real part, which implies there is an average transmitted wave going backwards, i.e the opposite direction of the incident wave.

Keywords: wave propagation, random media, multiple scattering, ensemble averaging

1 Introduction

Particulate materials consisting of particles that are randomly distributed within some homogeneous media are frequently used across many research disciplines and industries. For instance, these materials are used as contrast agents in medical ultrasound and other imaging fields. This is why accurate models of waves in these materials are of great importance.

When using light or sound waves to sense the particulates, it is necessary to understand how each particle scatters waves. In other words, it is important to consider that both particle properties and particle positions affect the total scattered waves, as shown in Figure (1). Although it is usually impossible to know both particle properties and positions, there are specific types of average measurements that can help us obtain reliable measurements for the total average wave $\langle u(x)\rangle$. These measurements are based on the ensemble averaging technique, which can both simplify the calculations and devise measurements that do not depend on the positions of the particles.

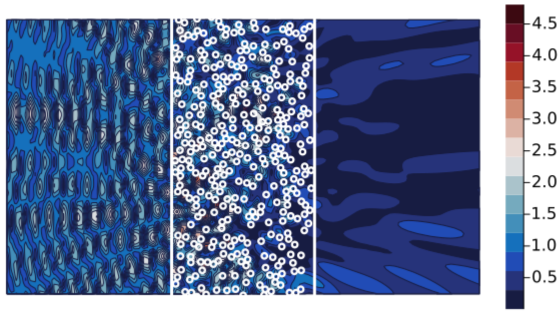

Figure 1: Scattering from one specific configuration of particles due to an incident plane wave striking a very large plate geometry filled with randomly distributed particles.

The process of calculating the ensemble average relates the speed of the effective wave c_* and the rate of attenuation α with the particles. The combination of those two measurables, forms the complex effective wavenumbers:

$$k_* = \frac{\omega}{c_*} + \mathrm{i}\alpha$$

where ω is the frequency.

In this paper, we look to design a computational experiment investigating the simplest case that shows the existence of at least two of these effective wavenumbers.

2 Why does it matter?

Two different theoretical models have predicted that there exist at least two (complex) effective wavenumbers for one fixed frequency [1]. These two wavenumbers would make a significant difference in the reflection and transmission [2] .

The theoretical models would predict the number of effective waves P described by the complex effective wavenumbers. This will allow us to obtain more accurate predictions for the overall average wave and by extension, the reflection and transmission coefficients. Hence, our objective is to verify these predictions using a robust numerical method based on high fidelity Monte Carlo simulations.

3 Theoretical and Numerical Predictions

Let us consider a plate filled with particles that are randomly distributed and a plane wave source that emits the incoming waves ($Fig.1$). As a result, the average transmitted wave inside the homogeneous, isotropic particulate material takes the form

$$\langle u(x) \rangle = \sum_{p=1}^{P} A_p \mathrm{e}^{\mathrm{i}k_p x - \mathrm{i}\omega t}. \tag{1}$$

In this expression, A_p represent the average transmission coefficient, k_p are the complex effective wavenumbers and x is a position vector inside the material. The factor $\mathrm{e}^{\mathrm{i}\omega t}$ is omitted for our convenience since it carries information about the incident wave emitted by the source.

Our goal is to approximate the A_p and k_p terms of the effective waves.

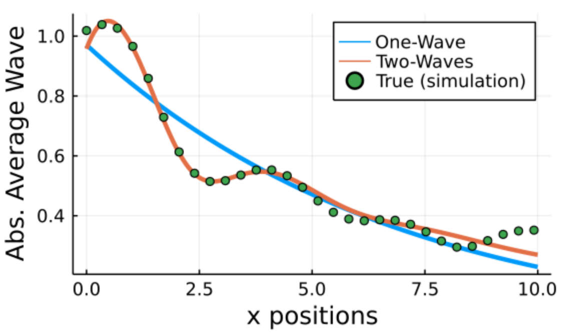

Figure 2: *True (simulation)* is the result of averaging over thousands of multi-pole simulations. *One-wave* is the current state-of-the-art mathematical model which assumes one effective plane-wave. *Two-waves* is from the multiple effective plane waves model comprising of two effective waves.

Recent studies show that in most cases, these effective waves are highly attenuating, meaning that the lower the attenuation $\operatorname{Im} k_p$, the greater the resulting amplitude of the effective wave and therefore the more it contributes to the overall average wave $\langle u(x) \rangle$.

Following our study case, we will use simpler methods to approximate only the first two coefficients A_1 and A_2 in equation (1) along with the incident plane-wave. Applying the asymptotic approximations, we get that

$$\langle u(x) \rangle = A_1 \mathrm{e}^{\mathrm{i}k_1 x} + A_2 \mathrm{e}^{\mathrm{i}k_2 x} + A_{in} \mathrm{e}^{\mathrm{i}k_{in} x} + \epsilon(x), \tag{2}$$

where $\epsilon(x)$ rapidly decays with x.

To extract the amplitude and the complex effective wavenumber of each effective wave forming equation (2), we employ the nonlinear least-square fitting, as shown in Figure (2). Data generated from high fidelity Monte Carlo simulations are fitted to a function that looks like equation (2). This leads to the desired coefficients A_1, A_2, k_1, k_2. At this point, we even discovered an effective wavenumber with a negative real part. In this peculiar case, the average transmitted wave propagates backwards, i.e. in the opposite direction to the incident wave.

Predicting the existence of more than one effective wavenumbers, increases the number of effective waves P. Hence, to produce more accurate results for the reflection and transmission, we need to generate phase diagrams that would indicate the essential number of effective waves.

In this talk, I will present a comparison between the fitted wavenumbers and the wavenumbers predicted by the Monte Carlo simulations. I will also discuss the necessity of the phase diagrams and the importance of calculating the wavenumbers for a wide range of frequencies, volume fractions and particle properties.

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