One-equation coupling of Curvilinear Virtual Element and Boundary Element methods for the wave equation in unbounded domains

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Abstract

We consider here wave propagation problems in bi-dimensional unbounded domains and, for their numerical solution, we propose the one-equation coupling of a Curvilinear Virtual Element Method (CVEM) with a Boundary Element Method (BEM). In particular, for the approximation in space, we consider decoupled approximation orders for the interior CVEM and the collocation BEM. For the approximation in time, we apply a time marching Crank-Nicolson scheme in the interior domain with the Lubich time convolution quadrature (CQ) formulas on the boundary. Exploiting the high order flexibility of the CVEM, the overall method allows us to use a low order BEM to retrieve accurate discrete solutions. Numerical tests show the effectiveness of the proposed approach.

Keywords: CVEM-BEM, one-equation coupling, convolution quadrature formula

1 Introduction

Let $\Omega_0 \subset \mathbb{R}^2$ a bounded domain with Lipschitz boundary Γ_0. We consider the damped wave equation in the unbounded domain $\Omega_e := \mathbb{R}^2 \backslash \overline{\Omega}_0$

$$\frac{1}{c^2}\ddot{u}_e(\mathbf{x};t) + \alpha\dot{u}_e(\mathbf{x};t) - \Delta u_e(\mathbf{x};t) = f(\mathbf{x};t) \quad (1)$$

with proper initial data and a Dirichlet boundary condition prescribed on Γ_0, c and α being the speed propagation and the damping parameter. Among the most commonly used approaches to solve (1), the Boundary Element Method turns out to be an appealing one, since it reduces the problem dimension by one, requiring only the discretization of the obstacle boundary. Once the boundary density is retrieved, the solution of the original problem at each point of the exterior domain is obtained by a post-processing procedure, based on an accurate computation of boundary integrals. However, this procedure may result not efficient, especially when the solution has to be evaluated in a wide region surrounding the obstacle. As an alternative approach, we define a finite computational domain, and we apply a coupling of an interior domain method with a boundary one. In literature, various coupling strategies have been proposed and extensively studied, among which we mention the Costabel-Han and the Johnson-Nédélec. Here we choose the latter, known also as the one-equation coupling, that, not involving a boundary integral operator of hypersingular type, is cheaper and easier to implement.

2 The one-equation coupling

Aiming at determining the solution u_e of (1) in a bounded subregion of Ω_e, we introduce an artificial boundary Γ which defines a finite computational domain Ω, bounded internally by Γ_0 and externally by Γ. Reformulating the PDE in the unbounded residual domain, in terms of a boundary integral equation (BIE) on Γ, and taking into account compatibility and equilibrium conditions, we reformulate the original problem as the coupling of the variational formulation of the wave equation in the interior domain, with the BIE on Γ. This latter involves both u, restriction of u_e to Ω, and its external normal derivative $\lambda := \nabla u \cdot \mathbf{n}$, and reads

$$\frac{1}{2}u(\mathbf{x};t) + \mathrm{V}\lambda(\mathbf{x};t) + \mathrm{K}u(\mathbf{x};t) = 0$$

for $(\mathbf{x};t) \in \Gamma \times [0,T]$, V and K being the single- and double-layer integral operators associated to the wave equation, respectively. The BIE is discretized by means of a collocation BEM in space and the BDF2 Lubich CQ formula in time.

3 Curvilinear Virtual Element Method

We discretize the weak formulation of (1), in the interior domain, by applying a CVEM which, in the spirit of the Virtual Element Method, consists in choosing conforming finite dimensional subspaces of $H^1(\Omega)$, whose "virtual" basis functions are not explicitly known. However, using only appropriate degrees of freedom, it is possible to compute an approximate version of the bilinear forms associated with the weak formulation. To avoid the approximation of the geometry in case of curvilinear obstacles, we consider a curvilinear VEM to guarantee the optimal convergence rate of the numerical method.

4 Decoupled approximation orders

We denote by $k_\circ$ and k_∂ the approximation orders of the CVEM and the BEM, respectively, associated with the mesh diameters $h_\circ$ in Ω and h_∂ on Γ. Based on the analysis performed in [2, 3] for the Helmholtz and Poisson problems, we may conjecture that at each fixed time, the error in the $L^2(\Omega)$-norm is bounded by

$$\|u - u_h\|_{L^2(\Omega)} = O\big(h_\circ^{k_\circ+1}\big) + O\big(h_\partial^{k_\partial+2}\big) + O\big(\Delta_t^2\big),$$

Δ_t being the time discretization parameter. The numerical tests confirm the expected error bound. We remark that the possibility of decoupling the approximation orders allows us to use a high order CVEM and a low order BEM. This turns out to be a great advantage for the global scheme, the accurate computation of the boundary integrals being a well-known bottleneck of the BEM, especially when the associated approximation order increases.

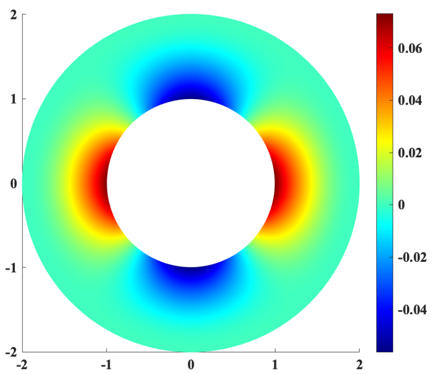

Figure 1: Solution of problem (1)-(2) at $T = 1$.

5 Numerical results

We consider equation (1) with null initial conditions and source term, $c = 1$, $\alpha = 0$, and Dirichlet datum

$$g(\mathbf{x}; t) = t^3 e^{-t^2} \cos(x_1^2 + 2x_2^2), \qquad (2)$$

in the region outside the unitary disk centered at the origin and, as artificial boundary, we choose the cirlce of radius 2. In Figure 1 we show the numerical solution obtained with orders $k_\circ = 2$, $k_\partial = 1$ in a mesh with 332,288 degrees of freedom, at the final time $T = 1$, with Δ_t =2.5e-03. In Figure 2 we report the convergence rate of the L^2-norm error for a fixed Δ_t with respect to space refinements. The optimal expected rate is reached for both choices of the discretization parameters.

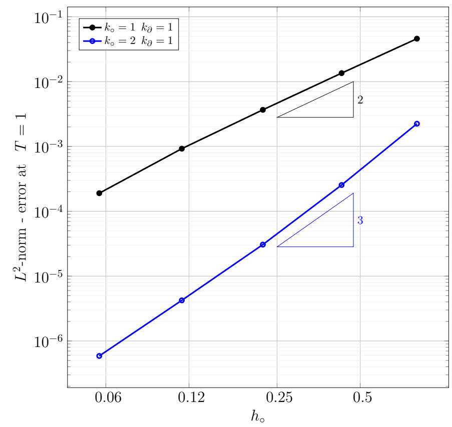

Figure 2: L^2-norm at $T = 1$, with fixed Δ_t.

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