A PML implementation for convex domains of general shape in time-harmonic acoustics

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Abstract

We address the efficient finite element solution of exterior acoustic problems with convex truncated computational domains of general shape surrounded by *perfectly matched layers* (PMLs). In this contribution, we will present a PML implementation that is versatile and automatic for the end-user. It relies on a mesh extrusion, a modification of the Jacobian matrix in the element-wise integrals and a parameter-free absorbing function. Only the PML thickness must be chosen. It will be validated and compared to other implementations using 2D and 3D cases.

Keywords: Finite elements, Helmholtz equation, Non-reflecting boundary condition, PML

1 Introduction

The PML is a popular absorbing boundary technique that combines accuracy, geometric flexibility and computational efficiency. In order to reduce the computational cost, it can be advantageous to minimize the size of the domain, leading to regions with general shapes.

Conformal PML formulations can address convex domains of general shape with smooth boundaries [3,5]. However, they depend on geometric parameters (*e.g.* principal curvatures and principal directions of the border), which may not be explicitly known if the geometry is complicated or if only the mesh is available.

In this short paper, we present a comprehensive implementation strategy, recently studied in [2] and based on [1], for Helmholtz problems. It is a specific implementation of the conformal PML for cases with smooth borders, but it can also be applied to cases with non-regular borders in an empirical way. After a presentation of the conformal PML, we present key aspects of our implementation and a numerical illustration.

2 Conformal PML

Let us consider a convex computational domain Ω_{dom} with a regular exterior border Γ. The layer Ω_{pml} is generated by extruding Γ in the normal direction $\mathbf{n}$ with a constant thickness.

To derive the PML equation, the Helmholtz equation is written in a local curvilinear coordinate system (ξ_1, ξ_2, ξ_3) associated to Γ. For each point $\mathbf{x}$ of the layer, the coordinate ξ_1 is the distance to the closest point $\mathbf{p}$ belonging to Γ, and the coordinates (ξ_2, ξ_3) are provided by a local parametrization of Γ at $\mathbf{p}$. After, the coordinate ξ_1 is replaced with the complex coordinate $\tilde{\xi}_1(\xi_1) := \xi_1 - f(\xi_1)/(\imath k)$, with $f(\xi_1) := \int_0^{\xi_1} \sigma(\zeta)\, d\zeta$, where $\sigma(\zeta)$ is the so-called absorbing function. See *e.g.* [3, 5].

By using standard techniques, we get the following variational formulation of the problem:

$$\left|\begin{array}{l} \text{Find } u \in H^1(\Omega) \text{ such that, for all } v \in H^1(\Omega), \\ \displaystyle\int_{\Omega_{\mathrm{pml}}} \Big[(\mathbf{J}_{\mathrm{pml}}^{-\top}\nabla_x u)\cdot(\mathbf{J}_{\mathrm{pml}}^{-\top}\nabla_x v) - k^2 uv\Big]\alpha_{\mathrm{pml}}\, d\Omega \\ \displaystyle + \int_{\Omega_{\mathrm{dom}}} \Big[\nabla_x u\cdot\nabla_x v - k^2 uv\Big] d\Omega = \int_{\Omega_{\mathrm{dom}}} fv\, d\Omega, \end{array}\right.$$

with $\alpha_{\mathrm{pml}} := \det(\mathbf{J}_{\mathrm{pml}})$ and the Jacobian matrix associated to the complex stretch,

$$\mathbf{J}_{\mathrm{pml}} := \mathbf{I} - \frac{1}{\imath k}\Big(\sigma(\xi_1)\,\mathbf{n}\mathbf{n}^\top + \sum_{j=2,3}\frac{\kappa_j f(\xi_1)}{1+\kappa_j\xi_1}\,\mathbf{t}_j\mathbf{t}_j^\top\Big),$$

where $\{\kappa_j\}_{j=2,3}$ are the principal curvatures and $\{\mathbf{t}_j\}_{j=2,3}$ are the principal directions of Γ.

3 PML implementation

The direct finite element implementation of the conformal PML requires the knowledge of the coordinate ξ_1 (which is a distance function) and the principal curvatures/directions of Γ at every point of the layer. We propose an implementation that provides all the required data.

Mesh extrusion and interpolation

The mesh of the layer is generated by extruding the mesh of the surface, Γ_h, along a direction $\mathbf{n}_h$ corresponding to the exterior normal. An empirical rule is proposed in [2] to deal with polyhedral surfaces. During this step, the distance function r_h, the direction $\mathbf{n}_h$ and the position of the closest point $\mathbf{p}_h$ are recorded at the nodes of the extruded mesh. Then, these nodal values are interpolated on every element of the layer by using polynomial basis functions. The interpolated fields verify $\mathbf{x}_h = \mathbf{p}_h + r_h\mathbf{n}_h$ everywhere, not only at the nodes.

Computation of element-wise integrals

The second key aspect of our approach is related to the computation of the element-wise integrals in the finite element matrix. For a given element D_e of Ω_{pml}, a typical integral reads

$$\int_{D_e} (\mathbf{J}_{\mathrm{pml},h}^{-\top}\nabla_x\Psi_A)\cdot(\mathbf{J}_{\mathrm{pml},h}^{-\top}\nabla_x\Psi_B)\,\alpha_{\mathrm{pml},h}\,dD_e,$$

where Ψ_A and Ψ_B are global basis functions. Using the mapping between the physical element D_e and the reference element D_{ref}, this integral can be rewritten as

$$\int_{D_{\mathrm{ref}}} (\mathbf{J}^{-\top}\nabla_u\psi_a)\cdot(\mathbf{J}^{-\top}\nabla_u\psi_b)\,(\det\mathbf{J})\,dD_{\mathrm{ref}},$$

where ψ_a and ψ_b are local basis functions that depend on the reference coordinates (u_1, u_2, u_3). The Jacobian matrix $\mathbf{J} := \mathbf{J}_{\mathrm{pml}}\mathbf{J}_{\mathrm{ref}}$ contains both the reference mapping and the complex stretch. The expression of $\mathbf{J}$ can be simplified by considering that the stretched coordinate actually is the interpolated distance function r_h. Then, the matrix $\mathbf{J}$ can be written explicitely as

$$\mathbf{J} = \mathbf{J}_{\mathrm{ref}} - \frac{1}{\imath k}\Big[\,(\partial_{u_1}r_h)\,\sigma(r_h)\,\mathbf{n}_h\ ;\ f(r_h)\,\partial_{u_2}\mathbf{n}_h\ ;\ f(r_h)\,\partial_{u_3}\mathbf{n}_h\,\Big].$$

This specific representation of the Jacobian matrix is the main novelty of this work.

Remarks

If the border of the domain is smooth, this approach is a specific implementation of the conformal PML with $\xi_1 \approx r_h$. Otherwise, it is an empirical approach that is rather good for edges and corners with obtuse angles.

A simple alternative [4] corresponds to using $\mathbf{J} = \big[\,\partial_{u_1}\tilde{\mathbf{x}}_h\ ;\ \partial_{u_2}\tilde{\mathbf{x}}_h\ ;\ \partial_{u_3}\tilde{\mathbf{x}}_h\,\big]$, where $\tilde{\mathbf{x}}_h$ corresponds to the stretched interpolated position vector. However, with this alternative, the function σ and f are interpolated, which introduce spurious errors if they are not polynomial.

We use a hyperbolic absorbing coefficient, $\sigma(r_h) = 1/(\delta - r_h)$, with the layer thickness δ. It provides good accuracy without requiring the tuning of free parameters, by contrast with polynomial functions.

4 Numerical illustration

We consider the scattering of a plane wave by a submarine. The border of the domain has been generated automatically with a convex hull algorithm. Simulations have been performed with P2 elements and PML thicknesses with 1, 5 and 10 mesh cells. The results are very close. More details and validation results are proposed in [2].

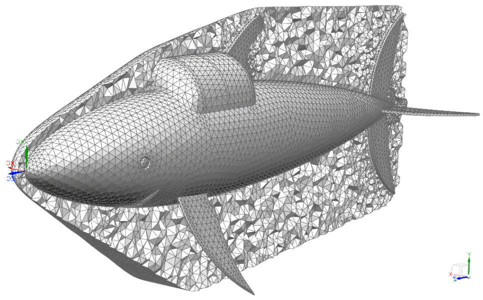

Figure 1: Mesh of the computational domain

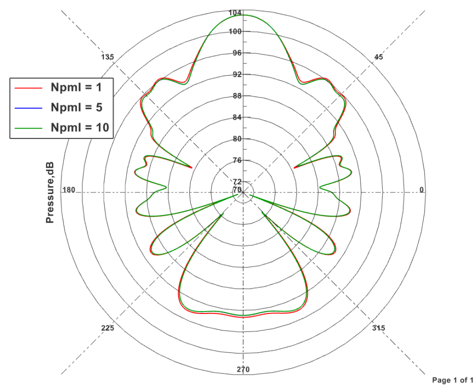

Figure 2: Directivity pattern of the scattered field for several PML thicknesses

References

[1] Bériot, Tournour (2009). On the locally-conformal PML imlement. for Helmholtz eqn. In *Inter-Noise/Noise-Con Conf. Proc.*, 503-513

[2] Bériot, Modave (2021), An automatic PML for acoustic FE simulations in convex domains of general shape. *IJNME*, 122:1239-1261

[3] Modave, Lambrechts, Geuzaine (2017). PMLs for convex truncated domains with DG time domain simulations. *CAMWA*, 73:684-700

[4] Ozgun, Kuzuoglu (2007). Non-Maxwellian locally-conformal PML absorbers for FE mesh truncation. *IEEE Trans. Ant. Prop.*, 55:931-937

[5] Teixeira, Chew (1998). Analytical derivation of a conformal perfectly matched absorber for EM waves. *Microw. Opt. Technol. Lett.*, 17:231-236