Stabilization of the high-order discretized wave equation for data assimilation problems

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Abstract

The objective of this work is to propose and analyze numerical schemes to solve data assimilation problems by observers for wave-like hyperbolic systems. The efficiency of the considered data assimilation strategy relies on the exponentially stable character of the underlying system. The aim of our work is therefore to propose a discretization process that enables to preserve the exponential stability at the discrete level when using high-order finite element approximation. The main idea is to add to the wave equation a stabilizing term which damps the oscillating components of the solutions (such as spurious waves). This term is built from a discrete multiplier analysis that gives us the exponential stability of the semi-discrete problem at any order without affecting the order of convergence.

Keywords: Data assimilation, Control, Numerical discretisation

1 Statement of the problem

We aim at studying data assimilation strategies for wave equation problems compatible with their high-order discretization. In sequential approaches, also called observer approaches [2], we aim at stabilizing exponentially fast – using the available measurements – the error between the observed trajectory initialized from an unknown initial condition and the simulated trajectory starting from a vanishing initial state.

Obtaining the exponential stabilization properties at the continuous level is widely studied – see for example [2]. However, discretizing the observer so that the stabilization property is preserved at the discrete level is an additional difficulty due to spurious high frequencies [1]. We propose, in the context of high order spectral finite elements schemes in space [3] and leap-frog discretisation in time, new remedies and associated analysis with a discretization - then - control strategy.

Let $\Omega = (0,1)$ be the domain of propagation, we consider a scalar wave equation with damping at the boundary $x = 1$,

$$\left\{\begin{array}{ll} \partial_t u + \partial_x v = 0 & \text{in } \Omega \\ \partial_t v + \partial_x u = 0 & \text{in } \Omega \end{array}\right. \qquad \left\{\begin{array}{l} v(0,t) = 0 \\ u(1,t) = \gamma v(1,t) \end{array}\right.$$

with $\gamma > 0$. The initial data corresponds to the unknown discrepancy between the observed trajectory (that is assumed observed at $x = 1$) and the reconstructed trajectory. By linearity, the system above corresponds to the system of the error between observed and reconstructed trajectory, and the question is to prove that such system – that is known to be exponentially stable – preserves its exponential stability property after discretization.

2 Eigenvalues behavior investigations

As a key indicator of the behavior of the discrete system we investigate the eigenvalues of the generator of the corresponding semi-group: we introduce

$$z = \begin{pmatrix} u \\ v \end{pmatrix}, \; A = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix},$$

where $A : L^2(0,1)\times L^2(0,1) \to L^2(0,1)\times L^2(0,1)$ is an unbounded operator with domain $D(A) \subset H^1(0,1) \times H^1(0,1)$ that takes into account the boundary conditions. The consider dynamics read

$$\dot{z} = Az, \quad \dot{z}_h = A_h z,$$

where $A_h \in \mathcal{L}(V_h)$ is an approximation of the operator A in a finite-dimensional space V_h that is a subspace of $\mathcal{D}(A)$ obtained using spectral-finite elements. Since we look for exponential stability we aim at constructing approximations for which the eigenvalues of A_h lies in the complex plane not close to the imaginary axis.

In Figure 1 is represented the spectrum of A_h obtained with a standard P_1 approximation (on both components u and v). We also represents the spectrum obtained when vanishing (with h) viscosity is added – as a classical stabilization strategy, see [1]. Only in the latter

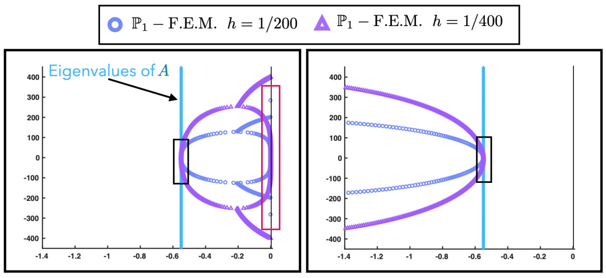

Figure 1: *Spectrum of A_h (left) and $A_h - h^2 V_h$ (right) (V_h is a discrete Laplace operator on both components). The black box highlights approximations of physical eigenvalues, the red one correspond to parasitic eigenvalues.*

case eigenvalues are well separated uniformly in h from the imaginary axis. The problem persists when using higher order finite elements and is in fact stronger since the classical stabilization strategy is less efficient, see Figure 2 and Figure 3 (left) in terms of quality of approximation.

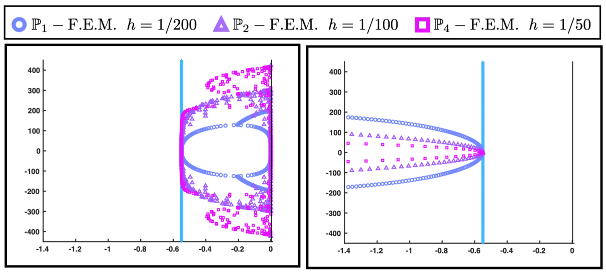

Figure 2: *Spectrum of A_h (left) and $A_h - h^2 V_h$ (right).*

3 A high-order stabilizing term

We use the following variational formulation of our problem : find $z_h(t) = (u_h(t), v_h(t)) \in V_h$ solution to, for every $(\tilde{u}_h, \tilde{v}_h) \in V_h$,

$$\left\{ \begin{aligned} & \oint_0^1 (\partial_t u_h \tilde{u}_h + \partial_t v_h \tilde{v}_h) \mathrm{d}x \\ & \quad + \int_0^1 (\partial_x u_h \tilde{v}_h + \partial_x v_h \tilde{u}_h) \mathrm{d}x \\ & \quad \quad + \tilde{v}_h(1)(\gamma v_h(1) - u_h(1)) \\ & \quad \quad \quad + d_h(u_h, \tilde{u}_h) + d_h(v_h, \tilde{v}_h) = 0, \end{aligned} \right.$$

where $\oint_0^1$ denotes the use of the Gauss-Lobatto quadrature formulae and d_h the correcting bilinear form used to obtained the desired stabilization property. Denoting $\{x_i\}$ the set of vertices of a partition of $[0, 1]$, it is defined by

$$d_h(u_h, \tilde{u}_h) = C_r h^{2r} \sum_i \int_{x_i}^{x_{i+1}} u_h^{(r)} \tilde{u}_h^{(r)} \mathrm{d}x,$$

where C_r is a positive scalar depending only on the order r of the finite-element method.

Denoting I_h the interpolation operator of continuous function into the finite element space, we use an original discrete multiplier strategy: choosing $\tilde{u}_h = I_h(x v_h)$ and $\tilde{v}_h = I_h(x u_h)$ in the formulation above we show that

$$\|z_h(t)\|_{L^2(\Omega)\times L^2(\Omega)} \leq C\, e^{\sigma t} \|z_h(0)\|_{L^2(\Omega)\times L^2(\Omega)},$$

where C and σ are positive scalars independent of h. This exponential stability property is confirm by an eigenvalue analysis (see Figure 3). Note that our method extends the standard stabilization strategy recovered here when $r = 1$.

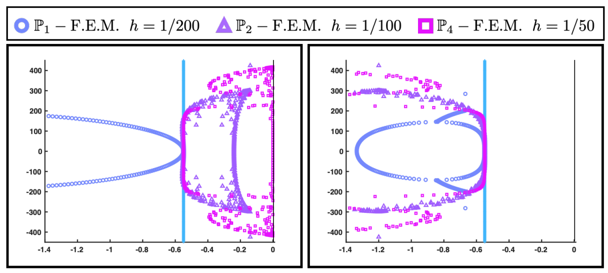

Figure 3: *Spectrum of $A_h - h^{r+1} V_h$ (left, r is the order of the method) (r is the order of the method) and $A_h - D_h$ (right). D_h is the operator constructed using the bilinear form d_h for each component.*

We complete our analysis by proposing and studying first an implicit then an explicit time discretization that are shown to preserve – under a time step condition – the efficiency, accuracy and exponential stability properties of the semi-discrete problem.

Extension to 2d problems and other choices of discretisation spaces will be discussed and numerical results will be presented.

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