Analysis and Approximation of Electromagnetic Surface Waves in Nonlinear Dispersive Media

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Abstract

We discuss well-posedness and stability results for nonlinear Maxwell equations, at an interface between dispersive media, based on evolutionary operator equations. Within this framework we propose a method for obtaining a justification of a wave packet approximation on long time intervals.

Keywords: Maxwell equations, evolutionary equations, dispersive waves

1 Maxwell system at an interface

Wave phenomena in nonlinear and interface optics are explained using the macroscopic Maxwell equations, where the material response (electric permittivity and magnetic permeability) is frequency-dependent, i. e., non-instantaneous. In physics, EM surface waves like *surface plasmon polaritons* at an interface are documented for a number of configurations. These are formal solutions of the linear Maxwell system and given by a plane-wave ansatz of the form

$$\varphi(x_1)\,\mathrm{e}^{i(kx_2-\omega t)}, \tag{1}$$

where $\varphi : \mathbb{R} \to \mathbb{R}^6$ is exponentially decaying, and $k \in \mathbb{R}$ and $\omega \in \mathbb{C}$ are related by a dispersion relation $\omega(k)$. These evenescent linear modes serve as building blocks for the approximation of solutions of the nonlinear Cauchy problem.

2 Evolutionary operator equations

The initial value problem for the Maxwell system is an evolutionary problem with memory and can be formulated as an operator equation

$$\partial_t M(\partial_t)u + \mathcal{A}u = F(u) + g \tag{2}$$

(in the sense of [1]) in the weighted Hilbert space

$$L^2_\varrho(\mathbb{R},\mathcal{H}) = \{u \in L^2_{\mathrm{loc}}(\mathbb{R},\mathcal{H}) : \|\mathrm{e}^{-\varrho t}u(t)\|_{L^2(\mathrm{d}t)} < \infty\}.$$

where $\mathcal{H} = L^2(\Omega)^6$. Here $\mathcal{A} = \left(\begin{smallmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl} & 0\end{smallmatrix}\right)$ is the Maxwell operator, ∂_t denotes the time derivative, and F is a (uniformly) Lipschitz continuous map on $L^2_\varrho(\mathbb{R},\mathcal{H})$. The operator $M(\partial_t)$ is called a linear material law and is related through the unitary Fourier-Laplace transform $\mathcal{L}_\varrho : L^2_\varrho(\mathbb{R},\mathcal{H}) \to L^2_0(\mathbb{R},\mathcal{H})$ to an analytic map $M : \mathbb{C}_{\mathrm{Re}>\varrho_0} \to \mathcal{B}(\mathcal{H},\mathcal{H})$ via $M(\partial_t) = \mathcal{L}_\varrho^* M(z)\mathcal{L}_\varrho$. The well-posedness of the (linear) problem, as well as properties like exponential stability, follow from (accretivity) conditions of the map $z \mapsto M(z)$, see [2,3].

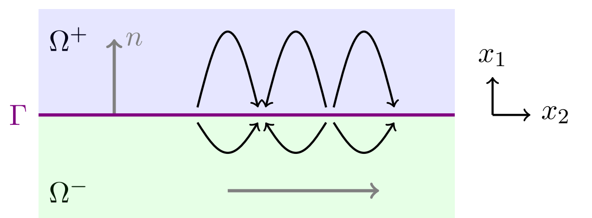

Figure 1: Schematic depiction of the electric field of a surface wave at the planar interface Γ between two media Ω^+ and Ω^- in $\mathbb{R}^3$.

3 Approximation of surface waves

Assume the setting in Figure 1 with $\Omega = \Omega^+ \sqcup \Gamma \sqcup \Omega^-$, let $\mathcal{H} = L^2(\Omega)^6$ and consider the linear material law

$$M(z) = M_0 + \frac{\alpha^\pm}{z} + \sum_{j=1}^{N} \frac{\beta_j^\pm}{z+\gamma_j^\pm},$$

on $\mathcal{H}$, where $\alpha^\pm > 0$, $\beta^\pm, \gamma^\pm \geq 0$ and M_0 is symmetric and positive definite. There exists $\nu_0 > 0$ such that $\operatorname{Re} z > -\nu_0 \implies \operatorname{Re} zM(z) \geq c$ holds for some $c > 0$. Thus, M satisfies the condition of exponential stability of the linear system $(\partial_t M(\partial_t) + \mathcal{A})u = g$ from [3], i. e., the solution operator $(\partial_t M(\partial_t) + \mathcal{A})^{-1}$ is bounded and causal on $L^2_\varrho(\mathbb{R},\mathcal{H})$ for large $\varrho > 0$, and, for small $\nu < \nu_0$, maps $L^2_{-\nu}(\mathbb{R},\mathcal{H})$ into itself. The question whether exponential stability can be expected also for the nonlinear system (2) can be answered in part by imposing local Lipschitz continuity of F (with small Lipschitz constant on small sets) in $L^2_{-\nu}(\mathbb{R},\mathcal{H})$ and using a fixed-point argument on a small ball.

Now let Φ_k for fixed k denote a linear mode as in (1). We model a wave packet propagating in x_2-direction by the multiple-scale ansatz

$$u_\varepsilon(t,x) = \varepsilon\, a(\varepsilon^2 t, \varepsilon(x_2 - c_{\mathrm{g}}t), \varepsilon^2 x_3)\Phi_k(t,x),$$

where $0 < \varepsilon \ll 1$ is a small parameter and a is a complex-valued amplitude. From (2) we obtain an equation for the error $R = u - u_\varepsilon$ of a similar form,

$$(\partial_t M_\varepsilon(\partial_t) + \mathcal{A})R + \mathrm{Res}(u_\varepsilon) = F_\varepsilon(R) + \tilde{g}, \quad (3)$$

where $\mathrm{Res}(u_\varepsilon) = (\partial_t M(\partial_t) + \mathcal{A})u_\varepsilon - F(u_\varepsilon)$, with the linear material law M_ε depending on u_ε, and where F_ε is nonlinear. Our aim here is to obtain a small global solution of (3) by applying the previous argument, which in turn justifies a long-time approximation of solutions of the initial Maxwell system (2). We give an outline of the conditions needed.

(a) Exponential stability of the linearized system. The necessary condition can be (for small ε) inherited from the material law M.

(b) Local Lipschitz-continuity of F_ε in the space $L^2_{-\nu}(\mathbb{R},\mathcal{H})$ with small Lipschitz constant. An example can be provided by a fully nonlocal model.

(c) Smallness of the residual, $\mathrm{Res}(u_\varepsilon) = o(\varepsilon)$ in $L^2_{-\nu}(\mathbb{R},\mathcal{H})$. Expanding into powers of ε, formally $\mathrm{Res}(u_\varepsilon) = O(\varepsilon^4)$ can be achieved through refinement of the ansatz, and demanding that a is a solution of an amplitude equation of complex Ginzburg-Landau type. Rigorous estimates can be obtained by imposing higher regularity on a, yielding $\mathrm{Res}(u_\varepsilon) = O(\varepsilon^{3/2})$ in $L^2_{-\nu}(\mathbb{R},\mathcal{H})$.

A fixed-point argument in $L^2_{-\nu}(\mathbb{R},\mathcal{H})$ finally yields small solutions R of (3) for small data $\tilde{g}$.

4 Future work

Electro-magnetic surface waves are often treated in the non-magnetic setting, where the magnetic permeability is constant. In this case the linear material law does not meet the requirements for exponential stability. Still, similar results can be obtained for the Maxwell system on a bounded domain [4].

References

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