Obtaining nonlinear acoustic models from non-conventional variational principles in fluid mechanics

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Abstract

Starting from various variational formulations of compressible viscous flows, linear and weakly nonlinear equations of acoustic wave propagation are derived and analysed in terms of additional terms compared to the classical wave equations. On the one hand, the focus is on extensions of the classical theory in the direction of viscous flows beyond the local thermodynamic disequilibrium and on small scales up to the limits of the continuum hypothesis; on the other hand, relativistic flows are considered which appear on very large scales. For the latter, the modelling of viscosity in accordance with the causality principle is still a subject of current debates, so that the analysis of acoustic wave propagation can make a valuable contribution to this issue. **Keywords:** Variational calculus, discontinuous Lagrangian, non-equilibrium Thermodynamics, weakly nonlinear wave equation, causality

1 Variational principles for fluid flow

The formulation of physical theories with variational principles enables a deeper understanding of the physical system in many respects. It can equally serve as a basis for the development of efficient solution methods. Two different approaches for fluid flow are considered below.

1.1 Clebsch potential approach

For compressible viscous flow without thermal conductivity, [1] proposed the Lagrangian:

$$\ell = -\varrho\left[\mathrm{D}_t\Phi + \alpha\mathrm{D}_t\beta + \frac{1}{\omega_0}\Im\left(\bar{\chi}\mathrm{D}_t\chi\right) - \frac{\vec{u}^{\,2}}{2} + e\right] + \frac{1}{2\mathrm{i}\omega_0}\ln\sqrt{\frac{\bar{\chi}}{\chi}}\,\underline{D}:\underline{R}\,, \tag{1}$$

where $e = e\left(\varrho, s\right)$ is the specific inner energy of the fluid depending on the mass density ϱ and $s = c_{p0}\ln\left(\bar{\chi}\chi/c_{p0}T_0\right)$ the specific entropy, given in terms of the complex thermal excitation χ [2], with temperature T_0, mass density ϱ_0 and specific heat c_{p0} as reference quantities. $\vec{u}$ denotes the velocity field, $\mathrm{D}_t = \partial/\partial t + \vec{u}\cdot\nabla$ the material time derivative, and $\underline{D}:\underline{R}$ the contraction of the shear rate tensor $\underline{D}$ with the friction tensor $\underline{R}$, taking viscosity into account.

Two Striking features are the discontinuity due to the logarithmic term, $\ln\sqrt{\bar{\chi}/\chi}$, and the additional parameter ω_0, both being related to phenomena away from thermodynamic equilibrium and beyond the continuum hypothesis, in particular Brownian molecular motion. In this context, $2\pi/\omega_0$ can be interpreted as a thermodynamic relaxation time.

The occurrence of the Clebsch variables Φ, α and β as fundamental fields is due to the required Galilei invariance of the Lagrangian [1].

1.2 Tensor potential approach

If one demands Lorentz invariance in place of Galilei invariance, the following proposal for a Lagrangian, motivated by analogy to Maxwell's theory, turns out to be suitable [3]:

$$\begin{aligned}\ell = &\left[\left(nmc^2 + ne + p\right)u^\alpha u^\beta + R^{\alpha\beta}\right]\bar{a}_{\alpha\beta} - nu^\alpha\partial_\alpha\chi \\ &+ \frac{1}{2}\partial^\mu\bar{a}^{\alpha\beta}\partial_\mu\bar{a}_{\alpha\beta} - \partial_\alpha\bar{a}^{\alpha\beta}\partial^\mu\bar{a}_{\mu\beta} - \frac{1}{3}\partial_\alpha\bar{a}^{\alpha\beta}\partial_\beta\Phi \\ &- \left[nmc^2 + ne - 3p + R^\alpha_\alpha\right]\frac{\Phi}{6} + \frac{\partial^\mu\Phi\partial_\mu\Phi}{12}\,,\end{aligned} \tag{2}$$

with n the particle density, s the specific entropy, u^α the 4-velocity, χ a Lagrange multiplier, Φ a scalar potential and $\bar{a}_{\alpha\beta}$ a traceless symmetric tensor potential. Again $e = e(n, s)$ is the specific internal energy, $p = n^2\partial e/\partial n$ the pressure and $R^{\alpha\beta}$ the friction tensor taking viscosity and heat conduction into account. Both e and $R^{\alpha\beta}$ depend on the constitutive relationships chosen to underpin the fluid model. Einstein's summation convention is used consecutively.

2 Derivation of weakly nonlinear acoustic models

Starting from a theory of fluid flow, weakly nonlinear acoustic models are obtained following a set procedure. Usually the first step is the computation of the respective Euler-Lagrange equations from the given Lagrangian, followed by manipulations leading to the equations of motion. The next steps refer to the case of classical flows considered in Sec. 1.1, but can also be applied to the relativistic case shown in Sec. 1.2 after necessary adaptations. As state equation for the pressure is assumed:

$$\frac{\partial p}{\partial \varrho} = a_0^2 \left[1 + \frac{B}{A}\frac{\varrho - \varrho_0}{\varrho_0}\right], \tag{3}$$

with small-signal sound speed a_0 and the nonlinearity parameter B/A of the respective fluid. Furthermore, sound waves are regarded as irrotational, $\nabla \times \vec{u} = \vec{0}$, implying:

$$\vec{u} = \nabla \Phi \,. \tag{4}$$

On introducing the condensation $\varepsilon := \ln(\varrho/\varrho_0)$ with equilibrium mass density ϱ_0, Taylor expansion w.r.t. ε and Φ up to terms of quadratic order while considering friction terms only in linear order, the set of equations can be reduced to one equation only for the potential [4]:

$$a_0^2 \left\{-\frac{1}{a_0^2}\partial_t^2 + \nabla^2\right\}\Phi = \tag{5}$$
$$\partial_t \left[(\nabla\Phi)^2 + \frac{B}{2Aa_0^2}(\partial_t\Phi)^2 - \bar{\nu}\left\{1 + \frac{\pi^2}{2\omega_0^2}\partial_t^2\right\}\nabla^2\Phi\right],$$

with $\bar{\nu}$ the diffusivity of sound over mass density. Eq. (5) is a generalisation of Kuznetsov's equation with an additional term due to thermodynamic non-equilibrium, the effect of which becomes visible in the dispersion relation [4]:

$$k = \frac{\omega}{a_0}\sqrt{\frac{1 + \mathrm{i}\frac{\bar{\nu}\omega}{a_0^2}\left(1 - \frac{\pi^2\omega^2}{2\omega_0^2}\right)}{1 + \left(\frac{\bar{\nu}\omega}{a_0^2}\right)^2\left(1 - \frac{\pi^2\omega^2}{2\omega_0^2}\right)^2}}\,, \tag{6}$$

resulting from the respective linearised equation. According to Fig. 1, the attenuation coefficient $\Im k(\omega)$ is smaller compared to the classical one.

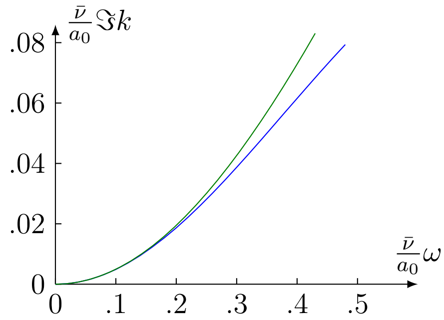

Figure 1: Attenuation coefficient $\Im k(\omega)$, plotted against angular frequency for the special choice $\omega_0 = 3a_0^2/\bar{\nu}$ (blue). For comparison, the respective relationship following from the classical theory ($\omega_0 \to \infty$) is shown (green).

3 Conclusions and perspectives

Deriving equations for acoustic wave propagation from unconventional fluid flow theories provides an excellent test scenario of such theories: while in case of the Lagrangian (1) thermodynamic non-equilibrium effects are identified as deviations from the classical theory, an analogue treatment of the relativistic Lagrangian (2) may help to test different constitutive models for the friction tensor $R^{\alpha\beta}$ in terms of causality. Also the role of potentials, for example the tensor potential $a^{\alpha\beta}$, and additional degrees of freedom such as the phase of the complex field χ, could be better understood via such models.

Another promising path could be the direct application of the procedure depicted in Sec. 2 to the Lagrangian density in order to formulate acoustic models by variational principles.

References

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