Modeling and analysis of an inverse boundary value problem in a two dimensional viscoelastic medium

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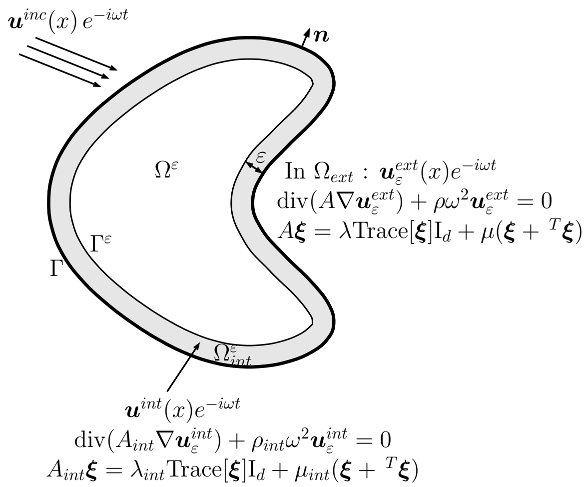
Abstract

We consider the solution of a transmission problem at a thin layer interface of thickness $\varepsilon > 0$ in a two dimensional homogeneous viscoelastic medium. A multi-scale expansion for that solution as ε tends to 0 enables to replace the thin layer with a generalized impedance boundary condition (GIBC) [3, 5]. This boundary condition involves a new second order surface symmetric operator with mixed regularity properties on tangential and normal components. Extending previous investigation for the Laplace equation case [2], the unique identification of the impedance parameters from measured data produced by the scattering of three linearly independent incident plane waves is established.

Keywords: linear viscoelasticity, thin layers, generalized impedance boundary conditions, inverse boundary value problem

1 Thin layer approximations

Let consider a simply connected bounded domain Ω in $\mathbb{R}^d$, $d = 2, 3$, with a closed orientable boundary Γ, as smooth as we need, representing a viscoelastic particle Ω^ε coated by a thin layer denoted Ω^ε_{int} with constant thickness $\varepsilon > 0$ and different material properties as below.



With a time-harmonic incident wave impinging upon Γ, the total wave is the solution of a transmission problem governed by the Navier equation in Ω_{ext} and Ω^ε_{int} with complex Lamé parameters characterized by $\mathrm{Re}(\lambda, \mu) > 0$ and $\mathrm{Im}(\omega\lambda, \omega\mu) \leq 0$. The continuity of the solution across the external boundary Γ is described by the following boundary conditions on Γ

$$\boldsymbol{u}^{ext}_\varepsilon + \boldsymbol{u}^{inc} = \boldsymbol{u}^{int}_\varepsilon$$
$$\boldsymbol{T}(\boldsymbol{u}^{ext}_\varepsilon + \boldsymbol{u}^{inc}) = \boldsymbol{T}\boldsymbol{u}^{int}_\varepsilon$$

where $\boldsymbol{T}\boldsymbol{u} = [A\nabla\boldsymbol{u}]\boldsymbol{n}$. This PDE system may be augmented by either a Dirichlet ($\boldsymbol{u}^{int}_\varepsilon = 0$) or a Neumann ($\boldsymbol{T}\boldsymbol{u}^{int}_\varepsilon = 0$) boundary condition on the internal boundary Γ^ε. In this context, the use of boundary and finite elements methods fails since some numerical instabilities arise due to different mesh scaling inside and outside the thin layer. To avoid the phenomenon, we approximate the original model by a new exterior boundary value problem with GIBCs.

In the first part of the talk, we present the asymptotic analysis [3, 5] leading to an approximate solution with error estimates up to $O(\varepsilon^2)$. The GIBC involves a new nonnegative symmetric second order differential operator whose mapping properties enable the extension of previous investigation [2] to linear viscoelasticity when $d = 2$. The regularity properties of the approximate solution are summarized in Section 2.

2 Regularity of the approximate solution

Let consider incident plane waves defined for a given unit direction $\boldsymbol{d}$ and polarization $\boldsymbol{p}$ by

$$\boldsymbol{u}^{inc}(x) = e^{i\kappa_s x\cdot\boldsymbol{d}}\boldsymbol{d}^\perp(\boldsymbol{p}\cdot\boldsymbol{d}^\perp) + \tfrac{1}{\lambda+2\mu}e^{i\kappa_p x\cdot\boldsymbol{d}}\boldsymbol{d}(\boldsymbol{p}\cdot\boldsymbol{d}).$$

The approximate problem we consider can be formulated as follows: Given an incident plane wave $\boldsymbol{u}^{inc}$ which is assumed to solve the Navier equations in the absence of any scatterer, find the scattered field $\boldsymbol{u}$ satisfying

$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\mathrm{div}\boldsymbol{u} + \rho\omega^2\boldsymbol{u} = 0 \quad \text{in } \Omega^c,$$

where the positive constant ρ is the density, and the impedance-like boundary condition on Γ

$$\boldsymbol{T}\boldsymbol{u} + i\omega\Big\{\alpha\boldsymbol{u} - \mathrm{div}_\Gamma\left(\beta(\mathrm{div}_\Gamma\boldsymbol{u})\mathrm{I}_{\boldsymbol{\tau}}\right)\Big\} = \boldsymbol{g},$$

with

$$\boldsymbol{g} = -\boldsymbol{T}\boldsymbol{u}^{inc} - i\omega\Big\{\alpha\boldsymbol{u}^{inc} - \mathrm{div}_\Gamma\left(\beta(\mathrm{div}_\Gamma\boldsymbol{u}^{inc})\mathrm{I}_{\boldsymbol{\tau}}\right)\Big\}.$$

where $\boldsymbol{\tau} = \boldsymbol{n}^\perp = {}^{\mathrm{T}}(-n_2, n_1)$, $\mathrm{I}_{\boldsymbol{\tau}} = \boldsymbol{\tau}\otimes\boldsymbol{\tau}$ and $\mathrm{div}_\Gamma\boldsymbol{u} = \boldsymbol{\tau}\cdot(\partial_s\boldsymbol{u})$ where ∂_s is the arc lentgh derivative. The impedance parameters $\alpha \in \mathscr{C}^1(\Gamma)$ and $\beta \in \mathscr{C}^2(\Gamma)$ are complex functions with non negative real parts. Moreover, the scattered field has to satisfy the Kupradze radiation condition rewritten in the form

$$\lim_{|x|\to+\infty} |x|^{\frac{1}{2}}\left|\boldsymbol{T}(\partial,\hat{x})\boldsymbol{u} - i\mathcal{K}_\omega\boldsymbol{u}\right| = 0,$$

which has to be satisfied uniformly for all unitary directions $\hat{x} = \frac{x}{|x|}$ and where

$$\mathcal{K}_\omega = \kappa_p(\lambda+2\mu)\mathrm{I}_{\hat{x}} + \kappa_s\mu\mathrm{I}_{\hat{x}^\perp},$$

$\kappa_p^2 = \omega^2\rho/(\lambda+2\mu)$ and $\kappa_s^2 = \omega^2\rho/\mu$ are the square P- and S-wave numbers associated to longitudinal and transverse wave propagation, respectively, such that if $\mathrm{Im}(\lambda,\mu) = 0$ then $\kappa_p = \omega\sqrt{\frac{\rho}{\lambda+2\mu}}$ and $\kappa_s = \omega\sqrt{\frac{\rho}{\mu}}$, or else $\mathrm{Im}(\kappa_p,\kappa_s) > 0$.

Lemma 1 *The surface differential operator* $\mathrm{div}_\Gamma\big((\mathrm{div}_\Gamma\cdot)\mathrm{I}_{\boldsymbol{\tau}}\big)$ *is bounded from* $\boldsymbol{H}^{\frac{3}{2}}(\Gamma)$ *to*

$$\boldsymbol{V}^{-\frac{1}{2}}(\Gamma) := \boldsymbol{H}_{\boldsymbol{\tau}}^{-\frac{1}{2}}(\Gamma) \oplus \boldsymbol{H}_n^{\frac{1}{2}}(\Gamma)$$

where $\boldsymbol{H}_\tau^{-\frac{1}{2}}(\Gamma) := \{\varphi \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma);\ \boldsymbol{n}\cdot\varphi = 0\}$ *and* $\boldsymbol{H}_n^{\frac{1}{2}}(\Gamma) := \{\varphi \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma);\ \boldsymbol{\tau}\cdot\varphi = 0\}$.

Theorem 2 *(Existence) Assume that* $|\beta| > 0$ *and* $\rho\omega^2$ *is not a Dirichlet eigenvalue of the Navier operator* $-\Delta^*$. *Then for each* $\boldsymbol{g} \in \boldsymbol{V}^{-\frac{1}{2}}(\Gamma)$, *the generalized impedance scattering problem admits one and only one solution in* $\boldsymbol{H}^2_{loc}(\overline{\Omega^c})$.

3 An inverse boundary value problem

In the second part of this talk, we present uniqueness results for the inverse problem of identifying the impedance functions α and β, when the shape Γ is known, from a finite number of far field patterns defined by $\boldsymbol{u}^\infty = \boldsymbol{u}_s^\infty + \boldsymbol{u}_p^\infty$ where

$$\boldsymbol{u}(x) = |x|^{-\frac{1}{2}}\left(e^{i\kappa_s|x|}\boldsymbol{u}_s^\infty(\hat{x}) + e^{i\kappa_p|x|}\boldsymbol{u}_p^\infty(\hat{x}) + O(\frac{1}{|x|})\right),$$

uniformly in all directions $\hat{x} = \frac{x}{|x|}$. The injectivity of the map $\alpha \mapsto \boldsymbol{u}^\infty$ obtained for the classical impedance acoustic problem in [2, Proposition 1] extends to linear (visco)elasticity as is. The case $\beta \neq 0$ is more involved.

For $j = 1,2,3$, we set $\boldsymbol{u}_j^{tot} = \boldsymbol{u}_j + \boldsymbol{u}_j^{inc}$. We introduce the Wronskian associated to the tangential components of the linear system

$$\begin{cases} \chi_j\boldsymbol{u}_j^{tot} + \chi_k\boldsymbol{u}_k^{tot} &= 0 \\ \chi_j\partial_s\boldsymbol{u}_j^{tot} + \chi_k\partial_s\boldsymbol{u}_k^{tot} &= 0, \end{cases}$$

given by

$$W_{j,k} = (\boldsymbol{u}_j^{tot}\cdot\boldsymbol{\tau})(\mathrm{div}_\Gamma\boldsymbol{u}_k^{tot}) - (\boldsymbol{u}_k^{tot}\cdot\boldsymbol{\tau})(\mathrm{div}_\Gamma\boldsymbol{u}_j^{tot}).$$

Hypothesis H1. The boundary curve Γ is such that given three linearly independent incident plane waves, the tangential components of the corresponding three total fields are linearly independent on Γ too, namely for $j,k = 1,2,3$ we have $W_{j,k} \neq 0$.

We prove the following results.

Theorem 3 *For a given shape* Γ, *under Hypothesis* ***H1***, *three far field patterns corresponding to the scattering of three incident plane waves, with linearly independent couples of directions and polarizations* $(\boldsymbol{d}_j, \boldsymbol{p}_j)_{j=1,2,3}$, *uniquely determine the impedance functions* α *and* β.

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