Symmetric-hyperbolic conservation laws modelling viscoelastic flows

Sébastien BOYAVAL[1,*]

[1]Laboratoire d'hydraulique Saint-Venant (Ecole des Ponts ParisTech – EDF R&D), EDF'lab Chatou & MATHERIALS, Inria Paris ; France. (sebastien.boyaval@enpc.fr)

*Email: me@my.domain

Abstract

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1 Problem setting

Many equations have been proposed to model flows with a viscoelastic behaviour, for various applications (polymer suspensions, turbulent fluids averaged in time/space…). Seminal equations have been proposed by Maxwell in 1867 for viscoelastic fluids, with stress relaxation [3]. In particular, the Upper-Convected Maxwell (UCM) equations are useful *for one-dimensional flows.* But for multi-dimensional flows, the usability of such viscoelastic fluid systems as the UCM equations remains limited.

For many configurations useful e.g. in materials engineering, where the viscoelastic fluid rheology is often invoked, numerous numerical simulations have shown that reasonable multi-dimensional extensions of the UCM equations do not converge when the discretization parameters are refined *beyond a critical value for the relaxation-time of the stress* [4].

2 A new model

As a remedy to the computations with existing equations, we propose to consider a new system of conservation laws with algebraic source terms (balance laws) to model viscoelastic flows:

$$
\begin{aligned}
&\partial_t(\rho u^i) + \partial_j \left(\rho u^j u^i\right) \\
&\quad - \partial_j \left(-p\delta_{ij} + 2\rho(K F^i_\alpha A^{\alpha\beta} F^j_\beta - \theta\delta_{ij})\right) = \rho f^i \\
&\partial_t(\rho F^i_\alpha) + \partial_j \left(\rho u^j F^i_\alpha - \rho u^i F^j_\alpha\right) = 0 \\
&\partial_t \rho + \partial_i(u^i \rho) = 0 \\
&\partial_t(\rho A^{\alpha\beta}) + \partial_j \left(\rho u^j A^{\alpha\beta}\right) \\
&\quad = \frac{4\rho}{\xi}\left(\theta\left([F^{-1}]^\alpha_i [F^{-1}]^\beta_i\right) - K A^{\alpha\beta}\right)
\end{aligned}
\tag{1}
$$

The system (1) is an extension of the elastodynamics with an additional material variable, the anelasticity tensor $A^{\alpha\beta}$, that is symmetric definite positive and relaxes at a time rate ξ to the inverse Cauchy-Green deformation tensor. Note that in (1), we used classical notations:

- u^i are the components of the velocity field in the Euclidean ambiant space,
- F^i_α are the components of the deformation gradient relative to the material manifold,
- $\rho = |F|^{-1}\rho_0$ is the mass density, and
- $p(\rho)$ is a pressure term.

We have denoted K and θ two constants, to be fixed in an isothermal setting.

Then, it is noteworthy that the Eulerian system (1) has a Lagrangian that is equivalent for smooth flows, and *symmetric-hyperbolic* [1]. Classically, the proof in [1] consists in exhibiting a *mathematical entropy* for (1) that is strictly convex in the conservative variable.

Moreover, the extended elastodynamics system (1) unifies hardly-elastic fluid models with hardly-compressible solid models, similarly to the famous K-BKZ integral viscoelastic models [4], but in a more versatile (purely differential) way based on an evolution equation for the anelasticity metric tensor.

3 Perpsectives

Contrary to the Navier-Stokes equations (i.e. momentum equations with velocity diffusion), our equations can model the viscous friction using *shear waves* of finite-speed.

The new system can be manipulated for various applications of the viscoelastic flow concept in environmental hydraulics (shallow-water flows) [1] or materials engineering (non-isothermal flows) [2].

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4 References

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