Coupled Single-Trace Formulations with Volume Integral Operators for Acoustic Transmission Problems

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Abstract

We study acoustic transmission problems in two dimensions with a bounded inhomogeneity. By means of defining reference constant coefficients, a representation formula for the exterior and interior domains is derived. The latter contains a volume integral operator, related to the one in the Lippmann-Schwinger equation. Following the approach of first and second-kind single-trace formulations (STF), a block operator is obtained and discretized. Numerical experiments confirm the convergence of the method and show its potential for high-contrast problems and scatterers with small inhomogeneities.

Keywords: integral equations, volume integral operators, boundary integral operators, single-trace formulations, acoustic transmission problems.

1 Introduction

We are interested in solving the acoustic wave transmission problem in presence of an inhomogeneous medium of compact support $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\Gamma := \partial\Omega$. Material properties are given by functions $a : \mathbb{R}^2 \to \mathbb{R}$ and $\kappa : \mathbb{R}^2 \to \mathbb{C}$ where

$$a(\mathbf{x}) \equiv 1, \kappa(\mathbf{x}) \equiv \kappa_0 \in \mathbb{C} \tag{1}$$

for $\mathbf{x} \in \mathbb{R}^2 \setminus \Omega$. The equation governing the problem of finding the total wave $u^{\text{tot}} := u + u^{\text{inc}}$ in the whole space is

$$-\text{div}(a(\mathbf{x})\nabla u^{\text{tot}}(\mathbf{x})) - \kappa(\mathbf{x})^2 u^{\text{tot}}(\mathbf{x}) = 0, \tag{2}$$

for $\mathbf{x} \in \mathbb{R}^2$, where u is the scattered field that satisfies radiation conditions

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - i\kappa_0 u\right) = 0, \tag{3}$$

and u^{inc} corresponds to an incident field that satisfies

$$-\Delta u^{\text{inc}}(\mathbf{x}) - \kappa_0^2 u^{\text{inc}}(\mathbf{x}) = 0, \tag{4}$$

for $\mathbf{x} \in \mathbb{R}^2$.

2 Volume Integral Equations

We can rewrite (2) as

$$-\Delta u^{\text{tot}} - \kappa_0^2 u^{\text{tot}} = \text{div}(\alpha\nabla u^{\text{tot}}) + \beta u^{\text{tot}}, \tag{5}$$

where $\alpha(\mathbf{x}) := a(\mathbf{x}) - 1$, $\beta(\mathbf{x}) := \kappa(\mathbf{x})^2 - \kappa_0^2$. The right-hand side of (5) is now a compactly supported function. Let N_0 denote the Newton potential for the Helmholtz equation with wavenumber $\kappa_0 \in \mathbb{C}$. It is possible to obtain from (5) the following integral equation, typically known as the Lippmann-Schwinger equation

$$u^{\text{tot}} - \text{div}N_0(\alpha\nabla u^{\text{tot}}) - N_0(\beta u^{\text{tot}}) = u^{\text{inc}}, \tag{6}$$

where $u^{\text{tot}} \in H^1(\Omega)$. Fredholmness of equation (6) has been studied in [2, 4], with remarkable limitations for the case $\alpha \in C^1(\Omega)$ instead of $C^1(\mathbb{R}^d)$.

3 Single-Trace Formulations

If we focus on the constant coefficients case, a useful approach to solve transmission problems requires boundary integral equations. Based on a representation formula for the interior and exterior domains

$$\begin{aligned} u &= S_1(\partial_n^- u) - D_1(\gamma^- u), && \text{in } \Omega, \\ u &= -S_0(\partial_n^+ u) + D_0(\gamma^+ u), && \text{in } \mathbb{R}^2 \setminus \Omega, \end{aligned}$$

where $\gamma^\pm, \partial_n^\pm$ denote exterior/interior Dirichlet and Neumann trace operators, S_j and D_j are the layer potentials for the Helmholtz equation with wavenumber $\kappa_j, j = 0, 1$. By taking traces on the representation formula, we obtain the following identities

$$\begin{aligned} \left(\frac{1}{2}I - A_1\right)\begin{pmatrix}\gamma^- u \\ \partial_n^- u\end{pmatrix} &= 0, \\ \left(\frac{1}{2}I + A_0\right)\begin{pmatrix}\gamma^+ u \\ \partial_n^+ u\end{pmatrix} &= 0 \end{aligned} \tag{7}$$

with A_j the Calderón operator with $\kappa_j, j = 0, 1$. By enforcing transmission conditions in (7), we obtain

$$\begin{aligned} \left(\frac{1}{2}I - A_1\right)\begin{pmatrix}\gamma^- u\\ \partial_n^- u\end{pmatrix} &= 0, \\ \left(\frac{1}{2}I + A_0\right)\begin{pmatrix}\gamma^- u\\ \partial_n^- u\end{pmatrix} &= \begin{pmatrix}\gamma u^{\mathrm{inc}}\\ \partial_n u^{\mathrm{inc}}\end{pmatrix}. \end{aligned} \tag{8}$$

First-kind STF is obtained by subtracting the expressions in (8):

$$(A_0 + A_1)\begin{pmatrix}\gamma^- u\\ \partial_n^- u\end{pmatrix} = \begin{pmatrix}\gamma u^{\mathrm{inc}}\\ \partial_n u^{\mathrm{inc}}\end{pmatrix}. \tag{9}$$

Second-kind STF is obtained by adding them:

$$(I + A_0 - A_1)\begin{pmatrix}\gamma^- u\\ \partial_n^- u\end{pmatrix} = \begin{pmatrix}\gamma u^{\mathrm{inc}}\\ \partial_n u^{\mathrm{inc}}\end{pmatrix}. \tag{10}$$

The appropriate functional setting for both equations has been previously studied in [1, 3].

4 Coupled STF-VIE

In the presence of a compactly supported source f in the right-hand side of the Helmholtz equation, the representation formula reads

$$u = S_1(\partial_n^- u) - D_1(\gamma^- u) + N_1(f), \tag{11}$$

for $\mathbf{x} \in \Omega$. We define reference coefficients

$$a_1 := \frac{1}{|\Gamma|}\int_\Gamma a(\mathbf{x})\mathrm{d}s_{\mathbf{x}}, \kappa_1 := \frac{1}{|\Gamma|}\int_\Gamma \kappa(\mathbf{x})\mathrm{d}s_{\mathbf{x}}. \tag{12}$$

This definition reduces the problem to the constant coefficient case of Section 3 when a and κ are constant functions. Following the approach from the STF and denoting

$$\mathcal{A}u := N_1\left(\mathrm{div}\left(\alpha_1 \nabla u\right) + \beta_1 u\right),$$

where $\alpha_1(\mathbf{x}) := a(\mathbf{x}) - a_1,\ \ \beta_1(\mathbf{x}) := \kappa(\mathbf{x})^2 - \kappa_1^2$, we can derive an extended version of the first-kind STF

$$\left(\begin{array}{c|c} M^{-1}A_0 M + A_1 & \begin{matrix}\gamma\mathcal{A}\\ \partial_n\mathcal{A}\end{matrix} \\ \hline -D_1 \quad S_1 & I - \mathcal{A} \end{array}\right)\begin{pmatrix}\gamma^- u\\ \partial_n^- u\\ u\end{pmatrix} = \begin{pmatrix}\gamma u^{\mathrm{inc}}\\ \partial_n u^{\mathrm{inc}}\\ 0\end{pmatrix}, \tag{13}$$

in $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \times H^1(\Omega)$, where

$$M = \begin{pmatrix}1 & 0\\ 0 & a\end{pmatrix}.$$

Similarly, an extended version of the second-kind STF can be derived.
These new formulations have some properties that make them desirable to be studied and compared with existing alternatives. We discuss:

(a) Well-posedness of the continuous and discrete formulations.

(b) Robustness of the formulation.

(c) Limit cases: boundary integral equations.

(d) Conditioning of the resulting matrix.

5 Numerical Experiments

Several numerical experiments illustrate the capabilities of our proposed formulations. We discretize with low-order finite elements on the boundary and in the domain. We study errors in the $L^2(\Omega)$ and $H^1(\Omega)$ norms.

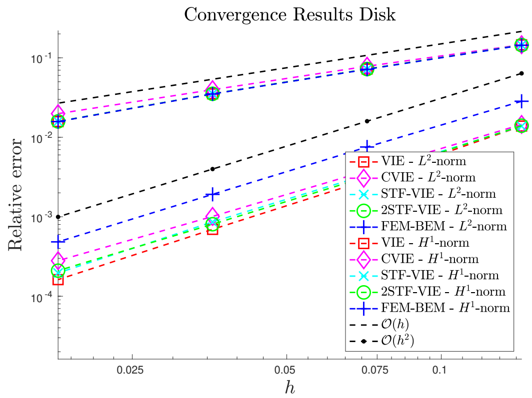

Figure 1: Scattering by a disk.

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