Wave power farm made of many rigid floating structures in Boussinesq regime

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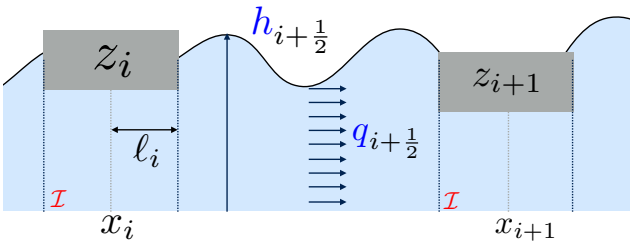
Abstract

This work deals with the interaction of waves governed by a Boussinesq system with some floating structures. The full system can be reduced to coupled boundary problems for the Boussinesq equations with boundary conditions given in terms of the vertical displacement of the objects, the average horizontal discharge beneath it and the traces of the water-column. The latter quantities are determined by nonlinear ODEs with forcing terms coming from the exterior wavefield.

Keywords: Fluid-structure interaction, Dispersive perturbation of hyperbolic problems, Wave Energy Converters in shallow water regime.

1 Modelling floating bodies



The mathematical study of floating structures is a keystone to understand wave power facilities. Consider N partially immersed rigid rectangles in a 2D fluid allowed to move vertically (see figure above). We denote by $2\ell_i$ and x_i the horizontal length and the abscissa of the center of the i-body. The domain of wave propagation $\mathbb{R}$ is divided into "congested" areas $\mathcal{I} = \cup_{i=1}^{N}(x_i - \ell_i, x_i + \ell_i)$ where the floating bodies are located and free areas $\mathcal{E} = \mathbb{R}/\mathcal{I}$. We will denote by $d_{i+\frac{1}{2}} = x_{i+1} - \ell_{i+1} - x_i - \ell_i$ the distance between the i-th body and $i+1$-th body.

The dynamics of each body is given by the Newton equation

$$m\ddot{z}_i + gz_i = \int_{x_i-\ell_i}^{x_i+\ell_i} \underline{P} \tag{1}$$

where the unknowns z_i denote the vertical displacement of the center of mass of the i-th body and $\ddot{\cdot}$ is the second derivative with respect to time. The source term $\underline{P}$ is the pressure exerted by the fluid under each body. At equilibrium the water column is piecewise constant

$$h_{\text{eq}} = \begin{cases} h_0 & \text{in } \mathcal{E} \\ h_0 - L_{w,i} & \text{in } (x_i - \ell_i, x_i + \ell_i) \end{cases}$$

where $L_{w,i}$ is the length of the immersed part of the vertical wall at equilibrium. The dynamics of the waves in $\mathbb{R}$ is described by the Boussinesq-Abbott system which is the following dispersive PDE

$$\begin{cases} \partial_t h + \partial_x q = 0 \\ (1 - \frac{h_{\text{eq}}^2}{3}\partial_x^2)\partial_t q + \partial_x\left(\frac{gh^2}{2} + \frac{q^2}{h}\right) = -h\partial_x \underline{P} \end{cases} \tag{2}$$

where q refers to the fluid horizontal discharge and h to the water-column. The latter is

- unknown in the free areas,
- should fit the wetted surface $h_{w,i} := h_{\text{eq}} + z_i - z_{\text{eq}}$ under any body $(x_i - \ell_i, x_i + \ell_i)$. In particular, this implies that

  $$q_{w,i} := q_{|(x_i-\ell_i, x_i+\ell_i)} = -(x - x_i)\dot{z}_i + \mathfrak{q}_i$$

  where the average is defined by

  $$\mathfrak{q}_i = \frac{1}{2}(q_{w,i}(x_i + \ell_i) + q_{w,i}(x_i - \ell_i)).$$

2 Pressure term

The pressure $\underline{P}$ is

- equal to the constant atmospheric one,
- unknown under any body and can be seen as Lagrangian multipliers $\partial_x \underline{P}_{|(x_i-\ell_i, x_i+\ell_i)}$ associated to the constraints $h = h_{w,i}$.

The system is not complete. Indeed one needs transmission conditions at each contact point $x_i \pm \ell_i$ between the free and "congested" areas.

The first transmission condition is associated to fluid volume conservation which implies that

$$q_{|\mathcal{E}}(x_i \pm \ell_i) = q_{|\mathcal{I}}(x_i \pm \ell_i) = \mp \ell_i \dot{z}_i + \mathfrak{q}_i.$$

The second one is associated to the conservation of energy of the full system. This is satisfied if we assume that

$$\underline{P} = g[h_{|\mathcal{E}} - h_{w,i}] + \frac{1}{2}\left(\frac{q^2_{|\mathcal{E}}}{h^2_{|\mathcal{E}}} - \frac{q^2_{w,i}}{h^2_{w,i}}\right) + \frac{h_0^2}{3}\frac{\ddot{h}_{|\mathcal{E}}}{h_{|\mathcal{E}}} - \frac{h^2_{\text{eq}}}{3}\frac{\ddot{z}_i}{h_{w,i}}$$

at each contact point $(x_i \pm \ell_i)$.

Taking the space derivative of the second equation in (2), one gets an elliptic equation for the pressure $\underline{P}$ on $(x_i - \ell_i, x_i + \ell_i)$ with Dirichlet boundary conditions and source depending on

$$\mathfrak{z}_i = (\mathfrak{q}_i, z_i, \dot{z}_i)$$

and its time derivative. If we use this elliptic problem in (1) and the second equation of (2) to eliminate $\underline{P}$, we get nonlinear ODEs

$$\mathfrak{m}(\mathfrak{z}_i, \underline{h}_i)\dot{\mathfrak{z}}_i + \gamma(\mathfrak{z}_i, \underline{h}_i) = \mathfrak{f}(\underline{h}_i, \ddot{\underline{h}}_i), \tag{3}$$

with forcing term $\mathfrak{f}(\underline{h}_i, \ddot{\underline{h}}_i)$ coming from the wave-field in free areas

$$\underline{h}_i = (h_{|\mathcal{E}}(x_i - \ell_i), h_{|\mathcal{E}}(x_i + \ell_i)).$$

The coefficients $\mathfrak{m}(\mathfrak{z}_i, \underline{h}_i)$, $\gamma(\mathfrak{z}_i, \underline{h}_i)$ and the forcing term $\mathfrak{f}(\underline{h}_i, \ddot{\underline{h}}_i)$ are similar to the ones given in [1]. The equations of wave-structure system are now complete.

3 An augmented formulation

Taking advantage of the dispersion terms, we can introduce a hidden equation for the traces of the water-column on each wall of floating solid. Firstly, we introduce the unknowns

$$u_{i+\frac{1}{2}}(x) = (h_{|\mathcal{E}}, q_{|\mathcal{E}})(x_i + \ell_i + d_{i+\frac{1}{2}}x)$$

that are cast on the unit cell $(0,1)$ for $i = 1...N-1$ and

$$\begin{cases} u_-(x) = (h,q)(x_1 - \ell_1 - x) \\ u_+(x) = (h,q)(x - x_N - \ell_N) \end{cases}$$

that are cast on $(0,\infty)$. Secondly, we introduce $\kappa_{i+\frac{1}{2}} := \frac{h_0}{\sqrt{3}} d_{i+\frac{1}{2}}$, the regularizing operators

$$R^{i+\frac{1}{2}} : H^n(0,1) \to H^{n+2}(0,1)$$

which are the inverse operators of $(1 - \kappa_{i+\frac{1}{2}}\partial_x^2)$ with homogeneous Neumann boundary conditions and R^0 the inverse operator of $(1 - \frac{h_0^2}{3}\partial_x^2)$ on $(0,\infty)$ with homogeneous Neumann boundary condition at $\{0\}$. Therefore, equations (2) can be rewritten on the conservative form on $(0,1)$

$$\partial_t u_{i+\frac{1}{2}} + d_{i+\frac{1}{2}}\partial_x \mathfrak{F}_{i+\frac{1}{2}} = \begin{pmatrix} 0 \\ \mathfrak{S}^-_{i+\frac{1}{2}} \end{pmatrix} e^{-\frac{x}{\kappa_{i+\frac{1}{2}}}} + \begin{pmatrix} 0 \\ \mathfrak{S}^+_{i+\frac{1}{2}} \end{pmatrix} e^{-\frac{1-x}{\kappa_{i+\frac{1}{2}}}} \tag{4}$$

and on $(0,\infty)$

$$\partial_t u_\pm \pm \partial_x \begin{pmatrix} q_\pm \\ R^0 f_\pm \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \dot{q}_\pm(0) e^{-x\frac{\sqrt{3}}{h_0}} \tag{5}$$

with boundary conditions

$$q_{i+\frac{1}{2}}(0) = -\ell_i \dot{z}_i + \mathfrak{q}_i, \quad q_{i+\frac{1}{2}}(1) = \ell_{i+1}\dot{z}_{i+1} + \mathfrak{q}_{i+1},$$
$$q_-(0) = \ell_1 \dot{z}_1 + \mathfrak{q}_1, \quad q_+(0) = -\ell_N \dot{z}_N + \mathfrak{q}_N.$$

In the previous equations, the flux terms are

$$\mathfrak{F}_{i+\frac{1}{2}} = \begin{pmatrix} q_{i+\frac{1}{2}} \\ R^{i+\frac{1}{2}} f_{i+\frac{1}{2}} \end{pmatrix} \quad f_{i+\frac{1}{2}} = \frac{g}{2}(h^2_{i+\frac{1}{2}} - h_0^2) + \frac{q^2_{i+\frac{1}{2}}}{h_{i+\frac{1}{2}}}$$

and the source terms are

$$\mathfrak{S}^-_{i+\frac{1}{2}} := \frac{(-\ell_i \ddot{z}_i + \dot{\mathfrak{q}}_i) - (\ell_{i+1}\ddot{z}_{i+1} + \dot{\mathfrak{q}}_{i+1})e^{-\frac{1}{\kappa_{i+\frac{1}{2}}}}}{1 - e^{-\frac{2}{\kappa_{i+\frac{1}{2}}}}}$$

$$\mathfrak{S}^+_{i+\frac{1}{2}} := \frac{(\ell_{i+1}\ddot{z}_{i+1} + \dot{\mathfrak{q}}_{i+1}) - (-\ell_i \ddot{z}_i + \dot{\mathfrak{q}}_i)e^{-\frac{1}{\kappa_{i+\frac{1}{2}}}}}{1 - e^{-\frac{2}{\kappa_{i+\frac{1}{2}}}}}.$$

Taking the spatial derivative of the second equation in (4), one gets a hidden equation on the water-column

$$\kappa^2_{i+\frac{1}{2}}\ddot{h}_{i+\frac{1}{2}} + f_{i+\frac{1}{2}} = R^{i+\frac{1}{2}} f_{i+\frac{1}{2}} + \kappa_{i+\frac{1}{2}}(\mathfrak{S}^-_{i+\frac{1}{2}} e^{-\frac{x}{\kappa_{i+\frac{1}{2}}}} - \mathfrak{S}^+_{i+\frac{1}{2}} e^{-\frac{1-x}{\kappa_{i+\frac{1}{2}}}}) \tag{6}$$

that can be taken at each contact point $x_i \pm \ell_i$ in order to be coupled with (3) to make an ODE in which the exterior wave-fields $R^{i+\frac{1}{2}} f_{i+\frac{1}{2}}$ acts as a forcing term. The full equations (3,6,4,5) are used to show local well-posedness theory and perform simulations with the finite volume method.

References

[1] G. Beck, D. Lannes, Freely Floating Objects on a Fluid Governed by the Boussinesq Equations. ⟨hal-03122615⟩