Probabilistic Wave Inversion Using Gibbs Posteriors: Application in Ultrasound Vibrometry

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Abstract

We propose a general framework for solving probabilistic wave inversion with applications in dispersion-based ultrasound vibrometry. Bayesian methods, while attractive, are often unable to handle the lack of a likelihood function due to the presence of nonlinear operators in the model. We adopt a loss-based framework [1] to estimate the parameter and summarize our confidence through *Gibbs posteriors.* Gibbs posteriors are derived as a solution to the variational problem on the space of distributional estimators of the parameter. We develop a cross-validation strategy that allows us to draw samples from the Gibbs posterior, tune the free parameter, and make pairwise comparisons between many models for practical inversion workflow. We support the merits of our method through simulated dispersion-based wave inversions that arise in the characterization of artertial vessels using ultrasound vibrometry.

Keywords: Uncertainty quantification, inverse problems, wave inversion, stochastic inversion

1 Introduction

The majority of Bayesian methods for inverse problems rely on an exact noise model, typically assumed to be i.i.d. Gaussian, to perform inference. It is desirable to extend such inference to more general settings where a noise model is unavailable or modeling the data generating mechanism is challenging. For instance, in arterial ultrasound vibrometry [2], space-time data is transformed to dispersion curves in the frequency domain through nonlinear operations. Then, these curves are used to invert for elastic or viscoelastic parameters. Even if the noise generating mechanisms are known in the space-time data, postulating closed-form likelihoods using dispersion curves becomes a very difficult (often impossible) task. The Gibbs posterior provides a way to update belief distributions in a general setting without the need of an explicit likelihood function [1]. Instead, the Gibbs posterior is applicable where the unknown parameters are only connected to the data through a loss function. Our contributions are: 1) the development of a principled approach to calibrating a Gibbs posterior that generalizes the Bayes posterior, and 2) the application of this framework to the inversion of geometric and material parameters in waveguides using dispersion data.

2 Incorporating Gibbs Posteriors to Inversion

Our interest is in solving an inverse problem for finite-dimensional parameter $\theta \in \Theta \subset \mathbb{R}^p$. Suppose we define a forward model through the following equation:

$$\mathcal{M}(\mathcal{F}(\theta), \theta) = 0. \tag{1}$$

$\mathcal{M}$ can be thought of as a PDE for an elastic medium, which implicitly defines the forward operator $\mathcal{F}$ that maps θ to a solution field of PDE. In practice, we solve the discretized equation model $\hat{\mathcal{M}}$ and compute a discretized solution field, $\hat{\mathcal{F}} : \Theta \to \mathbb{R}^D \times \mathbb{R}^T$. We assume observing an n-ensemble model (possibly $n > 1$) contaminated with an i.i.d. noise as

$$x_i(u,t) = \hat{\mathcal{F}}(\theta) + \epsilon_i,\ \epsilon_i \sim P_\epsilon,\ i = 1, \ldots, n. \tag{2}$$

We consider waveguide problems in the frequency domain. In this context, space-time states are transformed to dispersion curves in the frequency domain. Thus, we have the following observation model in $\mathbb{R}^d$:

$$y_i(\omega) = \hat{\mathcal{G}}(\hat{\mathcal{F}}(\theta) + \epsilon_i). \tag{3}$$

where $\hat{\mathcal{G}}$ is a nonlinear operator that maps space-time data to dispersion curves. We notice that the pushforward through $\hat{\mathcal{G}}$ complicates the simple linear noise model in 2.

The abstract model (3) usually has a nonlinear, non-smooth form, which leads to an intractable likelihood of the sample dispersion curves

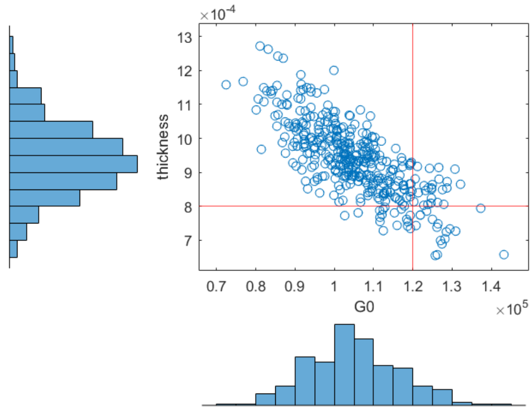

Figure 1: Posterior sample draws of shear modulus (G_0) and wall thickness learned from synthetic data against the truth (red).

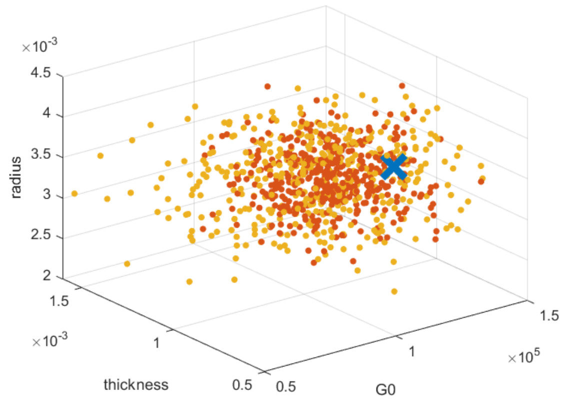

Figure 2: Posterior sample draws of arterial parameters produced using L^1 (orange) and L^2 loss (yellow) for simulated data and the truth (blue).

and a challenge to Bayesian methods. We instead assume an access to a loss function $l : \Theta \times \mathbb{R}^d \to \mathbb{R}$ quantifying discrepancy between the exact solution field $\mathcal{F}_d(\theta)$ and noisy samples y_i's. Then, we posit the following variational problem on the space of probability distributions for θ [1]:

$$\hat{\rho}(d\theta) = \arg\min_{\rho \ll \rho_0} \mathcal{L}(\rho), \tag{4}$$

$$\mathcal{L}(\rho) = W \sum_{i=1}^{n} \int l(\theta, y_i)\rho(d\theta) + D_{KL}(\rho || \rho_0). \tag{5}$$

The variational framework naturally extends the Tikhonov regularization in linear inverse problems, with W playing the role of a regularization parameter. The solution of the problem is the following Gibbs posterior for θ:

$$\hat{\rho}(d\theta) \equiv \frac{\exp(-W \sum_{i=1}^{n} l(\theta, y_i))\rho_0(d\theta)}{\int_\Theta \exp(-W \sum_{i=1}^{n} l(\theta, y_i))\rho_0(d\theta)}. \tag{6}$$

When a true likelihood model can be exactly specified, we can match W and loss function to that of a Bayesian posterior.

3 Computation and Simulation Studies

To develop a principled inversion workflow, we develop a cross-validation framework that simultaneously allows for drawing samples from a sequence of Gibbs posteriors and tuning W, using ideas from stochastic gradient descent (SGD) and sequential Monte Carlo (SMC). Furthermore, we show that a slight extension of the variational problem 6 and our calibration strategy allows for *between-model* comparison: for distinct loss functions, we can estimate their predictive optimality in the sample space $\mathcal{Y}$ using cross-validation.

We apply our methods to simulated data that mimic dispersion curves arising in dispersion-based wave inversions. The goal is confidence statement about the material and geometric parameters of arterial vessels using ultrasound vibrometry. Fig.1 shows that our Gibbs posterior estimate covers the truth used to simualte the data, with its shape capturing the correlation between the parameters due to the model structure. Fig.2 compares different Gibbs posterior samples obtained by using different loss functions. In this example, L^1 loss is similarly accurate as L^2 loss in covering the truth, but leads to a narrower confidence region estimate and better predictive performance.

4 References

References

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