Low-order Absorbing Boundary Conditions in HDG discretization of the convected Helmholtz equation

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Abstract

We describe how low-order boundary conditions for the convected Helmholtz equation can be constructed using the Lorentz transformation that maps the convected Helmholtz equation to the standard Helmholtz equation. The new ABCs are derived from the classical Bayliss-Turkel ABCs and are valid for a carrier flow that varies in the computational domain but is uniform in the exterior domain. They are easy to implement in an existing finite-element or discontinuous Galerkin solver and lead to accurate numerical results for both low and intermediate Mach numbers.

Keywords: Absorbing Boundary Conditions, convected Helmholtz equation, Lorentz transformation

1 Introduction

In many applications, the waves propagate in an infinite domain, which should be truncated in order to perform numerical simulations. Domain truncation is therefore an important part of computational wave dynamics and various techniques have been developed over the years. In this paper, we focus on the construction of low-order ABCs for the convected Helmholtz equation that can be easily implemented in a discontinuous Galerkin solver.

2 Convected Helmholtz equation

We consider the *convected Helmholtz equation* in an infinite domain

$$(-i\omega + \vec{v_0}\cdot\nabla)^2 p - \operatorname{div}\left(c_0^2\nabla p\right) = s, \quad \text{in } \mathbb{R}^N. \quad (1)$$

The finite computational domain is $\Omega \subset \mathbb{R}^N$, and we denote by Σ its boundary.

To construct new ABCs we make the following assumptions on the carrier flow:

(A-1) $\vec{v_0}$ is *incompressible*, *i.e.* $\operatorname{div}(\vec{v_0}) = 0$,

(A-2) $\vec{v_0}$ is *subsonic*, *i.e.* $|\vec{v_0}| < c_0$,

(A-3) $\vec{v_0}$ and c_0 are uniform in $\mathbb{R}^N\backslash\Omega$.

Assumptions (A-1) and (A-2) are used to ensure that (1) leads to a well-posed variational problem, whereas (A-3) is used to ease the construction of ABCs.

3 HDG discretization

Hybridizable Discontinuous Galerkin (HDG) methods are mixed DG methods that rely on a *static condensation* process to reduce the numerical cost. Using HDG methods allows to construct numerical solvers with all the advantages of DG methods (such as high-order, *hp*-adaptivity, natural parallelization,...) for a numerical cost similar to a continuous finite-element solver.

As detailed in [3], we need to rewrite (1) as a first-order system to construct a HDG method. We consider the following formulation

$$\vec{\sigma} + K_0\nabla p + 2i\omega p\vec{v_0} = 0, \quad (2)$$

$$-\omega^2 p + \operatorname{div}(\vec{\sigma}) = s, \quad (3)$$

where $K_0 = c_0^2\mathrm{Id} + \vec{v_0}\vec{v_0}^T$ and $\vec{\sigma}$ is the so-called *total flux.*

To work with the formulation (2)–(3), we want to construct an operator $\mathcal{Z}$ so that the ABC reads

$$\vec{\sigma}\cdot\vec{n} + \mathcal{Z}p = 0, \quad \text{on } \Sigma, \quad (4)$$

where $\vec{n}$ is a unit normal vector to Σ.

4 Lorentz transformation

Following [2], we introduce the frequency-domain Lorentz transform

$$\tilde{x} = Ax = \left(\mathrm{Id} + \frac{1}{\alpha(1+\alpha)}\vec{M_0}\vec{M_0}^T\right)x, \quad \tilde{\omega} = \frac{\omega}{\alpha}, \quad (5)$$

where $\alpha = \sqrt{1 - |\vec{M_0}|^2}$ is the *Lorentz factor* and $\vec{M_0} = \vec{v_0}/c_0$ is the *Mach vector.* For uniform carrier flows, this change of coordinates maps the convected Helmholtz equation to the standard one. More precisely, we have the

Theorem 1 *If $\vec{v}_0$ and c_0 are uniform and if $p(x,\omega)$ is a solution to* (1) *then*

$$\tilde{p}(\tilde{x},\tilde{\omega}) = \alpha \exp\left[i\frac{\omega}{\alpha^2 c_0}\vec{M}_0 \cdot x\right] p(x,\omega) \qquad (6)$$

is a solution to the standard Helmholtz equation

$$-\tilde{\omega}^2\tilde{p} - c_0^2\tilde{\Delta}\tilde{p} = \tilde{s}, \qquad (7)$$

where $\tilde{\Delta}$ is the Laplace operator in Lorentz coordinates.

5 Transformation of ABCs

We denote by $\tilde{\Sigma}$ the *artificial boundary* in Lorentz coordinates. As we will transform an ABC for $\tilde{p}(\tilde{x},\tilde{\omega})$ on $\tilde{\Sigma}$ into an ABC for p on Σ, it is convenient to chose $\tilde{\Sigma}$ circular, *i.e.*

$$\tilde{\Sigma} = \left\{\tilde{x} \mid |\tilde{x}|^2 = R^2\right\}. \qquad (8)$$

With this choice of $\tilde{\Sigma}$ the artificial boundary in physical coordinates is the following ellipse

$$\Sigma = \left\{x \mid |Ax|^2 = R^2\right\}. \qquad (9)$$

Then, we write an ABC for the standard Helmholtz equation (7) on $\tilde{\Sigma}$ as

$$\partial_{\tilde{n}} p + \tilde{\mathcal{Z}} p = 0, \text{ on } \tilde{\Sigma}. \qquad (10)$$

It can be transformed into an ABC for p using the

Theorem 2 *If $\vec{v}_0$ and c_0 are uniform in a neighborhood of Σ and if $\mathcal{Z}$ is defined as*

$$\mathcal{Z}(x,\omega) = -c_0^2|A^{-1}\vec{n}|\tilde{\mathcal{Z}}(\tilde{x},\tilde{\omega}) + i\omega\vec{v}_0\cdot\vec{n}, \qquad (11)$$

then

$$\vec{\sigma}\cdot\vec{n} + \mathcal{Z}p = 0, \text{ on } \Sigma, \qquad (12)$$

is an ABC for the convected Helmholtz equation.

Notice that the local uniformity of the carrier flow is ensured through assumption (A-3).

6 Numerical results

In **Table 1**, the error between the numerical solution and an analytic solution is given for various sizes of domains and for three ABCs: the transformation of the 0th and 1st order Bayliss-Turkel ABCs of [1], and an ABC that selects the outgoing plane waves that are locally orthogonal to Σ. Clearly (ABC-1) outperforms the two other ABCs. In **Figure 1** and **Figure 2**, we give two illustrative examples using (ABC-1).

R	He	ABC-0	ABC-1	ABC-PW
0.5	0.75	2.14%	0.67%	8.3%
1.0	1.5	1.21%	0.62%	7.31%
1.5	2.25	0.98%	0.66%	8.02%
2.0	3.0	0.83%	0.64%	7.1%

Table 1: Relative error in the domain for $|\vec{M}_0| = 0.6$

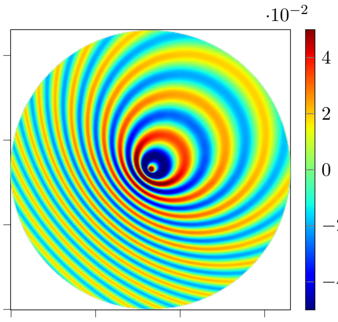

Figure 1: Point source in a uniform flow, $|\vec{M}_0| = 0.6$

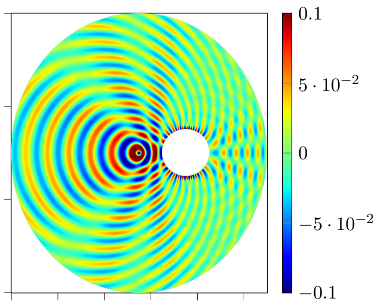

Figure 2: Point source in a potential flow, $|\vec{M}_0| = 0.4$

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