Energy method approach to existence results for the Helmholtz equation in periodic wave-guides

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Abstract

We consider the Helmholtz equation in an unbounded wave-guide and derive an existence result for non-singular frequencies. Proofs of such results exist, our emphasis is that our proof uses only energy methods. The flexibility of the new method allows to study also the case that two different media are used in the two unbounded directions

Keywords: Helmholtz equation, existence result, energy methods

1 Introduction

We study the Helmholtz equation

$$-\nabla \cdot (a\nabla u) = \omega^2 u + f \qquad \text{(H)}$$

in the wave-guide geometry $\Omega := \mathbb{R}\times S$ with $S \subset \mathbb{R}^{d-1}$ bounded Lipschitz. The coefficient of the operator is given by a function $a : \Omega \to \mathbb{R}$ which is strictly positive and 1-periodic in x_1. We impose a Neumann condition on $\partial\Omega$. The given data are a source term $f \in L^2(\Omega)$ with compact support and a frequency $\omega \in \mathbb{R}$; below we will assume some non-singularity assumption on ω. Our solution concept demands $u \in H^1_{\mathrm{loc}}(\Omega)$ and:

(i) u solves (H) in Ω in the weak sense

(ii) $\sup_{r\in\mathbb{Z}} \|u\|_{L^2((r,r+1)\times S)} < \infty$

(iii) a radiation condition is satisfied

The precise definition of the radiation condition (iii) is given below. Loosely speaking, (iii) on the right is demanding: There exist finitely many quasiperiodic homogeneous solutions φ_j of (H) with positive energy-flux and corresponding coefficients α_j such that $u - \sum_j \alpha_j\varphi_j \to 0$ as $x_1 \to \infty$. For further details on this condition and for an existence proof with other methods we refer to [1, 2].

Some further notation: We use the elliptic operator $Au := -\nabla\cdot(a\nabla u)$. Important bounded subsets of Ω are the cylinders $W_r := (r, r+1)\times S$, defined for arbitrary $r \in \mathbb{R}$. These cylinders allow, in particular, to introduce the norm $\|u\|_{sL} := \sup_{r\in\mathbb{Z}} \|u\|_{L^2(W_r)}$. To formulate our assumption on ω and to define below the radiation condition, we introduce the space of homogeneous solutions of the Helmholtz equation, $X := \{u|_{W_0} \,|\, u \in H^1_{\mathrm{loc}}(\Omega),\ \|u\|_{sL} < \infty,\ Au = \omega^2 u \text{ in } \Omega\}$.

We can now specify our assumption on the frequency.

Definition 1 (Non-singular frequency) $\omega > 0$ *is called a non-singular frequency for the coefficient* a *if:*

(a) Finite dimension: The space X *has a basis* $(\varphi_j)_{1\le j\le M}$ *with quasimoments* $\xi_j \in [0, 2\pi)$ *such that each* φ_j *possesses a* ξ_j*-quasiperiodic extension satisfying* $A\varphi_j = \omega^2\varphi_j$ *in* Ω.

(b) Non-vanishing flux: For every quasiperiodic function $u \in H^1_{\mathrm{loc}}(\Omega)$ *with* $Au = \omega^2 u$*, the restriction* $\varphi = u|_{W_0} \in X$ *has the property that the flux is non-vanishing:*

$$\Im \int_{W_0} a\nabla\varphi\cdot e_1\bar{\varphi} \neq 0\,.$$

The basis $(\varphi_j)_{1\le j\le M}$ can be improved to another basis $(\phi_j^\pm)_{1\le j\le N}$ with $2N = M$ with some orthogonality properties and with the property that the flux of ϕ_j^+ and ϕ_j^- is positive and negative, respectively. Since we have a basis of X, every $u \in X$ can be written as $u = \sum_{j=1}^N \alpha_j\phi_j^+ + \sum_{j=1}^N \beta_j\phi_j^-$ with appropriate factors $\alpha_j, \beta_j \in \mathbb{C}$. This allows to define projections, e.g., $\Pi_{X,+} : u \mapsto \sum_{j=1}^N \alpha_j\phi_j^+$ onto right-going waves. Together with the orthogonal $L^2(W_0)$-projection we can define projections onto right- and left-going waves

$$\Pi_\pm : L^2(W_0) \to X_\pm \subset L^2(W_0).$$

These projections allow to extract, from an arbitrary function $u \in L^2(W_0)$ the right-going part and the left-going part. The projections also allow to make the radiation condition precise. Our definition turns out to be equivalent with more classical definitions; our definition is useful since our proofs imply the radiation condition in this form.

Definition 2 (Radiation condition) *Let ω be non-singular and $\Pi_\pm$ the above projections. We say that $u : \Omega \to \mathbb{C}$ with $\|u\|_{sL} < \infty$ satisfies the radiation condition if*

$$\Pi_-(u|_{W_r}) \to 0 \text{ and } \Pi_+(u|_{W_{-r}}) \to 0 \text{ as } r \to \infty\,.$$

We identify a function on W_r with a function on W_0 via a shift.

2 Main results

As announced earlier, our results are based on energy methods. Loosely speaking, this means that we use only L^2-based function spaces and that our proof relies on testing procedures. A multiplication of the equation $Au = \omega^2 u$ with u or, more precisely, with the complex conjugate of u, and an integration over the domain $\{\rho < x_1 < r\}$ for arbitrary $-\infty < \rho < r < \infty$, we obtain

$$\Im \int_{\{\rho\}\times S} a\nabla u \cdot e_1\ \bar{u} = \Im \int_{\{r\}\times S} a\nabla u \cdot e_1\ \bar{u}\,.$$

The quantity on the left is the flux of u at position ρ; more precisely, it is the right-going energy flux. The above equality therefore expresses energy conservation: the total (energy-) flux into the domain $\{\rho < x_1 < r\}$ is vanishing.

Our results are based on energy conservation principles. Let us turn to the results.

Theorem 3 (Periodic media) *Let the data Ω, f, ω, a be as described, in particular, with the periodicity $a(x+e_1) = a(x)$ $\forall x \in \Omega$ and with the frequency ω being non-singular. Then there exists one and only one solution u to the radiation problem (i)–(iii).*

The method of proof decouples the problem on the left and on the right hand side. Because of this, we can also treat media that are periodic on the left and periodic on the right, but these could be two different periodic media. Our assumption on the medium can also be formulated as follows: There are two periodic fields $a^{\text{left}}, a^{\text{right}} : \Omega \to \mathbb{R}^{d\times d}$, $a^{\text{left}}(x+e_1) = a^{\text{left}}(x)$ and $a^{\text{right}}(x+e_1) = a^{\text{right}}(x)$ for every $x \in \Omega$. The coefficient a is of class $L^\infty(\Omega)$, it is pointwise symmetric and positive and has the ellipticity bounds $\Lambda > \lambda > 0$. It satisfies, for some $R_0 > 0$:

$$\begin{aligned} a(x) &= a^{\text{left}}(x) \qquad \text{if } x_1 < -R_0\,,\\ a(x) &= a^{\text{right}}(x) \qquad \text{if } x_1 > R_0\,. \end{aligned}$$

We obtain the following result, which takes the form of a Fredholm alternative.

Theorem 4 (Non-periodic media) *Let Ω be as above, let $a : \Omega \to \mathbb{R}^{d\times d}$ be periodic outside a compact set: For some $R_0 > 0$ holds*

$$a(x + e_1) = a(x)$$

for every $x \in \Omega$ with $|x_1| > R_0$. Let $\omega > 0$ be a non-singular frequency for the left and the right medium. If (i)–(iii) with $f = 0$ possesses only the trivial solution, then there is a unique solution u for arbitrary $f \in H^{-1}(\Omega)$ with compact support.

The proof for the above two theorems can be found in the preprint [3]. The principal idea is to solve a Helmholtz problem in a truncated domain $\{-R < x_1 < R\}$. This is done with a technique that was first used in [4], the solution $u = u_R$ is found with an application of the Lax-Milgram theorem. The energy method is used to derive uniform bounds for the sequence $(u_R)_R$. The limit $R \to \infty$ provides the desired solution u of the radiation problem.

References

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