A fast time-stepping method for the Westervelt equation with time-fractional damping

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Abstract

We consider the attenuated Westervelt equation, with the attenuation governed by a non-local in time operator. The non-locality is described by a time convolution with a singular kernel, the simplest case being that of the Riemann-Liouville fractional integral. We describe a time-stepping method and how a recently developed fast and memory efficient method for fractional derivatives can be applied to lessen the impact the non-locality has on the computational costs. Numerical results complete the work.

Keywords: Westervelt, damped wave equation, time-fractional derivative

1 Damped nonlinear wave equation

In this work we consider the damped Westervelt equation on a smooth domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$,

$$\partial_t^2 u - \Delta u - a\beta * \Delta\partial_t u = k\partial_t^2(u^2), \qquad (1)$$

where

$$f * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

denotes the one sided convolution, $a > 0, k > 0$ constants and $\beta : \mathbb{R}_+ \to \mathbb{R}_+$. Westervelt equation is a fundamental model in nonlinear acoustics with, e.g., the above damped equation capable of modelling of ultrasound in lossy media such as human tissue.

We assume that the kernel β satisfies

$$\beta \in L^1_{\text{loc}}(0,\infty), \quad \beta \geq 0, \quad \beta' \leq 0 \qquad (2)$$

and that its Laplace transform $\hat{\beta}$ satisfies

$$\operatorname{Re}\frac{1}{\hat{\beta}(z)} \geq C_\beta(\sigma) > 0 \qquad \text{for } \operatorname{Re} z \geq \sigma > 0. \qquad (3)$$

Typical example of β we have in mind is

$$\beta(t) = \frac{1}{\Gamma(\mu)} t^{\mu-1} e^{-rt}, \qquad (4)$$

for constants $r \geq 0$ and $\mu \in (0,1)$. If $r = 0$, then

$$\beta * f = I_t^\mu f \quad \text{and} \quad \beta * \partial_t f = \partial_t^{1-\mu} f,$$

where I_t^μ denotes the Riemann-Liouville fractional integral of order $\mu \in (0,1)$ and $\partial_t^{1-\mu}$ the Caputo derivative of order $1-\mu \in (0,1)$. The Laplace transform of β in this case is

$$\hat{\beta}(z) = (z+r)^{-\mu}.$$

Thus (3) is also satisfied as

$$\operatorname{Re}\frac{1}{\hat{\beta}(z)} \geq (\sigma + r)^\mu \qquad \text{for } \operatorname{Re} z \geq \sigma > 0.$$

A number of other non-local damping operators can be found in literature. These include combinations of fractional time-derivatives and fractional space derivatives. For a list and existence and uniqueness of solutions to resulting equations, see the recent [5].

We rewrite equation (1) as

$$\begin{aligned}(1-2ku)\partial_t^2 u - \Delta u - a\beta * \partial_t \Delta u(t) ds &= 2k(\partial_t u)^2 \\ u &= 0 \\ u(0) = u_0, \quad \partial_t u(0) &= u_1,\end{aligned}$$

where homogeneous Dirichlet condition is taken for simplicity. Next, we present a time-discretisation of the equations in this form.

2 Time-discretization

Let $\Delta t > 0$ be the uniform time-step and $t_n = n\Delta t$ the discrete times at which u_n denotes the approximation of $u(t_n)$. We introduce some notation to more easily present the discretization:

$$\begin{aligned} Du_n &= \frac{1}{2\Delta t}(u_{n+1} - u_{n-1}), \\ D^2 u_n &= \frac{1}{\Delta t^2}(u_{n+1} - 2u_n + u_{n-1}), \\ \{u\}_n &= \frac{1}{4}(u_{n+1} + 2u_n + u_{n-1}). \end{aligned}$$

We now discretize the equation using the trapezoidal rule (a second-order, A-stable Newmark scheme). The time-discrete system then reads:

$$(1 - 2k\{u\}_n)D^2u_n - \Delta\{u\}_n - a\beta *_{\Delta t} D\Delta u_n \\ = 2k(Du_n)^2,$$

where $\beta *_{\Delta t} g_n$ denotes a convolution quadrature [6] approximation of $\int_0^{t_n} \beta(t_n - \tau)g(\tau)d\tau$. Similar time-discretization but with explicit time-stepping was analyzed in [1] for the linear case $k = 0$.

Under the assumptions on β, convolution quadrature conserves a certain positivity property of the convolution, namely

$$\sum_{j=0}^{\infty} \varrho^{2j} \langle v_j, \beta *_{\Delta t} v_j \rangle \\ \geq C_\beta(\tilde{\sigma}) \sum_{j=0}^{\infty} \varrho^{2j} \|\beta *_{\Delta t} v_j\|_2^2,$$

for any $\varrho \in (0, 1)$; see [4]. If $\lim_{\sigma \to 0^+} C_\beta(\sigma) > 0$, the scaling parameter ϱ can be set to 1, which simplifies the technicalities involved in analysing the stability and convergence of the above semi-discrete system; see [2]. The latter is the case for β as in (4) with $r > 0$.

3 Oblvious computation of the memory

Convolution quadrature of the time-discretization has the form

$$\beta *_{\Delta t} v_n = \sum_{j=0}^{n} \omega_{n-j} v_j,$$

where ω_j are convolution weights, which in general decay only slowly as $j \to \infty$. Thus, the numerical scheme needs to keep $O(N)$ solution vectors u_j in memory where N is the number of time-steps and has computational time increasing quadratically with N. This makes realistic computation difficult or impossible. Recent fast and oblivious algorithm for convolution quadrature of fractional integrals presented in [3] is directly applicable in the case (4) with $r = 0$ and can reduce the memory requirements to $O(\log N)$ and computational cost to $O(N \log N)$. The case of $r > 0$ in (4) can also be dealt with the same algorithm with some small changes.

4 Numerical results

Numerical analysis of the time-discrete system is given in [2] including a number of numerical evidence supporting the analysis. Here we present some basic computational results in 1D. The parameters in (1) are set to $k = 0.09$, $a = 0.1$, and $\mu = 0.5$ and $r = 0$ in (4). The initial data is a Gaussian

$$u_0(x) = e^{-40x^2} \qquad v_0(0) = 0.$$

The spatial discretisation is performed by the piecewise linear Galerkin finite element method. The solution is plotted in Figure 1 at various times.

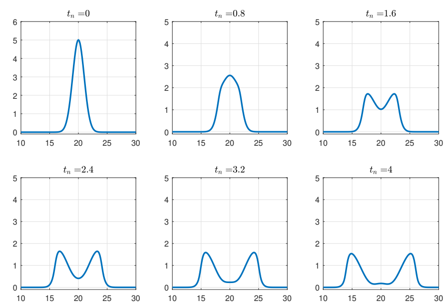

Figure 1: Solution of fractionally damped Westervelt in 1D.

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