# Effective dynamics for low-amplitude transient elastic waves in a 1D periodic array of non-linear interfaces

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**Abstract**

This presentation focuses on the time-domain propagation of elastic waves through a 1D periodic medium that contains non-linear imperfect interfaces, i.e. interfaces exhibiting a discontinuity in displacement and stress governed by a non-linear constitutive relation. In this context, we investigate transient waves with both low-amplitude and long-wavelength, and aim at deriving homogenized models that describe their effective motion.

**Keywords:** Homogenization – Correctors – Imperfect interfaces – Non-linear waves – Time-domain numerical simulations

## 1 Objectives

The array of interfaces considered is generated by a, possibly heterogeneous, cell repeated periodically and bonded by interfaces that are associated with transmission conditions of non-linear "spring-mass" type. More precisely, the imperfect interfaces are characterized by a linear dynamics but a non-linear elasticity law. The latter is not specified at first and only key theoretical assumptions are required. To establish an effective model, the two-scale asymptotic homogenization method is deployed, up to the first-order. To begin, an effective model is obtained for the leading zeroth-order contribution to the microstructured wavefield. It amounts to a wave equation with a non-linear constitutive stress-strain relation that is inherited from the behavior of the imperfect interfaces at the microscale. The next first-order corrector term is then shown to be expressed in terms of a cell function and the solution of a linear elastic wave equation. Without further hypothesis, the constitutive relation and the source term of the latter depend non-linearly on the zeroth-order field, as does the cell function. Combining these zeroth- and first-order models leads to an approximation of both the macroscopic behavior of the microstructured wavefield and its small-scale fluctuations within the periodic array.

## 2 Setting: microstructured configuration

We consider the propagation of transient waves in a 1D periodic elastic medium containing imperfect interfaces. The latter have spacing $h$ and, for simplicity but with no loss of generality, we consider that they are located at $X_n = nh$ with $n \in \mathbb{Z}$. The elastic medium is supposed to be $h$-periodic and linear elastic with mass density $\rho_h(X)$ and Young's modulus $E_h(X)$. Given a source term $F$, the displacement field $U_h$ is governed by the time-domain wave equation

$$\rho_h(X)\frac{\partial^2 U_h}{\partial t^2}(X,t) = \frac{\partial \Sigma_h}{\partial X}(X,t) + F(X,t) \quad (1)$$

where

$$\Sigma_h(X,t) = E_h(X)\frac{\partial U_h}{\partial X}(X,t),$$

with $\Sigma_h$ being the stress field. Moreover, the interfaces are assumed to be characterized by the interface *mass* and *rigidity* parameters $M$ and $K$, respectively, together with the, possibly *non-linear*, constitutive relation $\mathcal{R}$, so that the following transmission conditions apply at any interface point $X_n$, see [1–4]:

$$\begin{cases} M\left\langle\!\!\left\langle \dfrac{\partial^2 U_h}{\partial t^2}(\cdot,t)\right\rangle\!\!\right\rangle_{X_n} = [\![\Sigma_h(\cdot,t)]\!]_{X_n} & \text{(2a)}\\ \langle\!\langle \Sigma_h(\cdot,t)\rangle\!\rangle_{X_n} = K\,\mathcal{R}\left([\![U_h(\cdot,t)]\!]_{X_n}\right), & \text{(2b)}\end{cases}$$

where, for any function $g(X)$, we define the jump and mean operators $[\![\cdot]\!]_{X_n}$ and $\langle\!\langle\cdot\rangle\!\rangle_{X_n}$ as

$$\begin{aligned} [\![g]\!]_{X_n} &= g(X_n^+) - g(X_n^-),\\ \langle\!\langle g\rangle\!\rangle_{X_n} &= \frac{1}{2}\big(g(X_n^+) + g(X_n^-)\big). \end{aligned} \quad (3)$$

In addition, both the displacement $U_h$ and the stress field $\Sigma_h$ are continuous on the open intervals $(X_n, X_{n+1})$.

## 3 Main homogenization results

We now consider a reference wavelength $\lambda^*$ and introduce the following parameters

$$k^* = \frac{2\pi}{\lambda^*} \qquad \text{and} \qquad \eta = hk^*, \tag{4}$$

$k^*$ being the reference wavenumber. In this study it is assumed that $\eta \ll 1$ and that the source term $F$ is of relatively *low-amplitude* (an issue that will be discussed). The objective is to derive an effective dynamical model, up to the first-order, for the waves propagating in the periodic interface array considered. More precisely, we seek an approximation $U^{(1)}$ of the solution $U_h$ to (1–2) of the form:

$$U_h(X,t) = U^{(1)}(X,t) + o(h).$$

The main results of this study is that the sought-after approximation is given by

$$U^{(1)}(X,t) = U_0(X,t) + hU_1(X,t), \tag{5}$$

where the zeroth-order field $U_0$ in (5) is continuous and is solution of the problem

$$\rho_{\text{eff}} \frac{\partial^2 U_0}{\partial t^2}(X,t) = \frac{\partial \Sigma_0}{\partial X}(X,t) + F(X,t)$$

with

$$\Sigma_0(X,t) = \mathcal{G}_{\text{eff}}\big(\mathcal{E}_0(X,t)\big).$$

Here, $\mathcal{E}_0 = \partial U_0/\partial X$ and $\mathcal{G}_{\text{eff}}$ is an effective strain-stress relation that is local and, generally speaking, non-linear, while $\rho_{\text{eff}}$ is an effective mass density. Moreover, the first-order corrector field $U_1$ in (5) can be written as

$$U_1(X,t) = \overline{U}_1(X,t) + \mathcal{P}\big(y, \mathcal{E}_0(X,t)\big)\, \mathcal{E}_0(X,t)$$

with $y = (X - nh)/h$ for $X \in \big(nh, (n+1)h\big)$ and where the *cell function* $\mathcal{P}$ is, generally speaking, a non-linear function of $\mathcal{E}_0$. The *mean* field $\overline{U}_1$ is solution to the *linear* problem:

$$\rho_{\text{eff}} \frac{\partial^2 \overline{U}_1}{\partial t^2}(X,t) = \frac{\partial \overline{\Sigma}_1}{\partial X}(X,t) + \mathcal{S}\big(U_0(X,t)\big)$$

with

$$\overline{\Sigma}_1(X,t) = \mathcal{G}'_{\text{eff}}\big(\mathcal{E}_0(X,t)\big)\, \overline{\mathcal{E}}_1(X,t),$$

where $\overline{\mathcal{E}}_1 = \partial \overline{U}_1/\partial X$, while both the *parameter* $\mathcal{G}'_{\text{eff}}\big(\mathcal{E}_0(X,t)\big)$, which is the derivative of $\mathcal{G}_{\text{eff}}$, and the source term $\mathcal{S}\big(U_0(X,t)\big)$ depend explicitly on the zeroth-order field, locally in space and time, and in a non-linear fashion.

Particularizing for a prototypical non-linear interface law and in the cases of a homogeneous periodic cell and a bilaminated one, the behavior of the obtained models will then be illustrated on a set of numerical examples and compared with full-field simulations. Both the influence of the dominant wavelength and of the wavefield amplitude will be investigated numerically, as well as the characteristic features related to non-linear phenomena.

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