High-frequency homogenisation in periodic media with imperfect interfaces and elastodynamic co-dipole metaclusters

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Abstract

This work will tackle two independent issues that are of interest in the framework of metamaterials which take advantage of a specific structure of the material in order to control the propagation of waves at a macroscopic scale. Metamaterials are usually divided into two categories: periodic media at high frequency usually named phononic crystals and locally resonant metamaterials that allow the occurence of bandgaps at lower frequencies and with no need of periodicity.

(i) In a first part, the concept of high-frequency homogenisation, which is relevant in the framework of phononic crystals, is extended to the case of one-dimensional periodic media with imperfect interfaces of the spring-mass type.

(ii) The framework of the second part is local resonances for non-periodic configurations in full elasticity. We study the manner by which the dipole resonance arising from a soft-in-rigid configuration is affected by coupling and geometry when several scatterers are in close proximity.

Keywords: (i) High-frequency homogenisation, Periodic media, Imperfect interfaces

(ii) Elastodynamic metamaterials, Dipole resonance, Metaclusters

1 High-frequency homogenisation in periodic media with imperfect interfaces

Phononic crystals are exactly described by the Floquet-Bloch theory that provides the exact relation of dispersion that can be solved numerically. This allows to describe the occurence of band-gaps that are intervals of frequency in which wave propagation is forbidden.

However, the wave fields associated with these short wavelengths can also present large evolution lengths and this information is not directly accessible by the Floquet-Bloch approach. The concept of high-frequency homogenisation [1–3] allows to study this question of large scale modulation of wave fields with short wavelengths.

This concept is extended to the case of one-dimensional periodic media with imperfect interfaces of the spring-mass type. In other words, when considering the propagation of elastic waves in such media, discontinuities are allowed across the borders X_n of the periodic cell for the displacement U_h and the stress $E_h \frac{\mathrm{d}U_h}{\mathrm{d}X}$ as follows:

$$[U_h]_{X_n} = \frac{1}{K} \left\langle E_h \frac{\mathrm{d}U_h}{\mathrm{d}X} \right\rangle_{X_n}, \qquad (1)$$

$$[E_h \frac{\mathrm{d}U_h}{\mathrm{d}X}]_{X_n} = -M\omega^2 \left\langle U_h \right\rangle_{X_n}, \qquad (2)$$

with ω the frequency, $[\cdot]$ and $< \cdot >$ the jump and the mean value across the inferface, and M and K the mass and stiffness characterizing the imperfect interface.

The homogenisation is carried out about the periodic and antiperiodic solutions corresponding to the edges of the Brillouin zone. Due to the discontinuities at the interfaces, the zeroth-order eigenvalue problem obtained is unusal since the eigenvalue also appears in the boundary conditions. This requires to introduce a tailored inner product to prove the symmetry and non-negativity of the associated operator.

Asymptotic approximations are provided for both the branches of the dispersion diagram (second order) and the resulting wave field (leading order, see Figure 1). Compared to the usual homogenisation, the main difference is that the macroscopic variable is the amplitude of the eigenmode associated to the edges of the Brillouin zone instead of the displacement field itself.

The case of two branches of the dispersion diagram intersecting with a non-zero slope at an edge of the Brillouin zone (occurrence of a so-called Dirac point) is also considered, result-

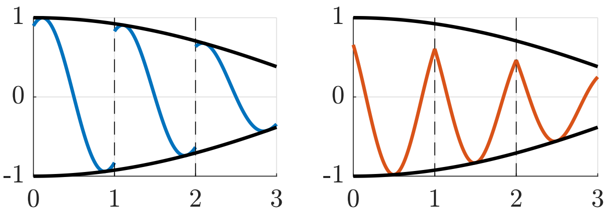

Figure 1: Zeroth-order wavefield near the lower (left) and upper (right) ,branch of the first band gap. Large scale modulation function in black.

ing in an approximation of the dispersion diagram (first-order) and the wave field (zeroth-order) near these points. Finally, a *uniform approximation* valid for both Dirac and non-Dirac points is provided.

Numerical comparisons are made with the exact solutions obtained by the Bloch-Floquet theory. Convergence measurements are carried out on the wave fields to validate the approach. For both the wave field and the dispersion relation (see Figure 2), we show that the uniform approximation remains a very good approximation even far from the edges of the Brillouin zone.

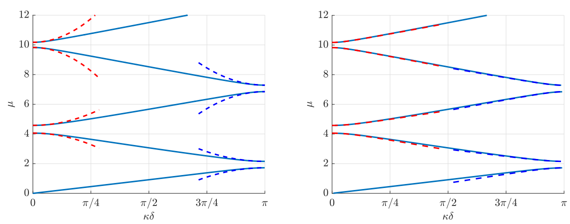

Figure 2: Exact dispersion diagram (plain lines) and homogenized approximations (dotted lines). (left): single approximation. (right): uniform approximation

2 Elastodynamic co-dipole metaclusters

In the framework of metamaterials, resonances are of great interest since they allow to reach a subwavelength response and do not require periodicity compared to phononic crystals. In elasticity, a dipole resonance was first studied in [4] in which a rigid core inclusion is placed in a softer layer of silicone rubber itself placed in a hard epoxy matrix.

The concept of metacluster introduced in [5] takes advantage of a collection of scatterers in order to tune the far-field response due to some incident field. In [6], a metacluster of voids leading to a monopole resonance, i.e. voids in an elastic matrix which has a bulk modulus much greater that its shear modulus, is studied and it is shown that the configuration of circular cylindrical voids can have a significant effect on the resonant frequency and the far-field response.

In the present work, we extend the latter study for coated cylinders of circular cross-section that present the physical contrasts of [4], and consequently give rise to a dipole resonance. More precisely, we study the manner by which the resonance arising from this soft-in-rigid configuration in close proximity is affected by coupling, i.e. multiple scattering. In particular, we show that by modifying the configuration of the metacluster, e.g. reducing the distance between them, the resonance can be enhanced. Therefore, the scattering amplitude and the directionality can be tailored and tuned playing on the geometry of the configuration. Such metaclusters could then be used in the design of elastodynamic metamaterials in order to control both wave propagation and effective properties.

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