Long time behaviour for electromagnetic waves in dissipative Lorentz media

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Abstract

A very general class of models for describing the propagation of waves in dispersive electromagnetic media is provided by generalized Lorentz models [1]. In this work, we study the long time behaviour of the solutions of the dissipative version of these models.

1 Introduction

We are interested in Maxwell's equations

$$\left\{\begin{array}{l} \varepsilon_0\, \partial_t\, \mathbf{E} - \nabla\times\mathbf{H} + \varepsilon_0\, \partial_t\, \mathbf{P} = 0 \\ \mu_0\, \partial_t\, \mathbf{H} + \nabla\times\mathbf{E} + \mu_0\, \partial_t\, \mathbf{M} = 0, \end{array}\right. \tag{1.1}$$

where $\mathbf{E}$ and $\mathbf{H}$ are, respectively, the electric and magnetic fields, while $\mathbf{P}$ (resp. $\mathbf{M}$) is the electric polarization (resp. magnetization). In generalized dissipative Lorentz media, these are related to the electromagnetic field via a system of ODE's (the constitutive laws of the medium)

$$\left\{\begin{array}{l} \mathbf{P} = \displaystyle\sum_{j=1}^{N_e} \Omega_{e,j}^2\, \mathbf{P}_j, \quad \mathbf{M} = \sum_{\ell=1}^{N_m} \Omega_{m,\ell}^2\, \mathbf{M}_\ell, \\ \partial_t^2\, \mathbf{P}_j + \alpha_{e,j}\, \partial_t\, \mathbf{P}_j + \omega_{e,j}^2\, \mathbf{P}_j = \mathbf{E}, \\ \partial_t^2\, \mathbf{M}_\ell + \alpha_{m,\ell}\, \partial_t\, \mathbf{M}_\ell + \omega_{m,\ell}^2\, \mathbf{M}_\ell = \mathbf{H}, \end{array}\right. \tag{1.2}$$

where $(\mathbf{P}_j, \partial_t\mathbf{P}_j, \mathbf{M}_\ell, \partial_t\mathbf{M}_\ell)$ vanish at $t = 0$. In (1.2), $(\Omega_{e,j}, \Omega_{m,\ell}) > 0$ while $(\alpha_{e,j}, \alpha_{m,\ell}) \geq 0$ and $(\omega_{e,j}, \omega_{m,\ell}) > 0$. Our goal is to analyze the long time behaviour of the solution of the Cauchy problem associated to (1.1, 1.2) and any initial data $(\mathbf{E}_0, \mathbf{H}_0)$ under the so-called weak dissipative assumption

$$\sum_{j=1}^{N_e} \alpha_{e,j} + \sum_{\ell=1}^{N_m} \alpha_{m,\ell} > 0. \tag{1.3}$$

2 Main results

In what follows $f \lesssim g$ means $f \leq C\, g$, for some constant $C > 0$ independent of $x, t, \mathbf{E}_0, \mathbf{H}_0$.

Let $\mathcal{E}(t)$ be the electromagnetic energy

$$\mathcal{E}(t) = \frac{1}{2} \int_{\mathbb{R}^3} \left(\varepsilon_0\, |\mathbf{E}|^2 + \mu_0\, |\mathbf{H}|^2\right)(x,t)\, dx \tag{2.1}$$

The generic result (cf. Remark 3) is as follows

Theorem 1 *For $\mathbf{E}_0$ and $\mathbf{H}_0$ in $L^2(\mathbb{R}^3)^3$ such that $\nabla\cdot\mathbf{E}_0 = \nabla\cdot\mathbf{H}_0 = 0$, the energy $\mathcal{E}(t)$ tends to 0 when $t \to +\infty$. Moreover if for some integers $m \geq 0$ and $p \geq 0$,*

$$\left|\begin{array}{l} (\mathbf{E}_0, \mathbf{H}_0) \in H^m(\mathbb{R}^3)^3 \times H^m(\mathbb{R}^3)^3, \\ |x|^p\, (\mathbf{E}_0, \mathbf{H}_0) \in L^1(\mathbb{R}^3)^3 \times L^1(\mathbb{R}^3)^3, \\ \int_{\mathbb{R}^3} x^\alpha\, \mathbf{E}_0\, dx = 0,\ \int_{\mathbb{R}^3} x^\alpha\, \mathbf{H}_0\, dx = 0,\ \forall\, |\alpha| < p \end{array}\right. \tag{2.2}$$

one has a polynomial decay rate

$$\mathcal{E}(t) \leq \frac{C_r^m(\mathbf{E}_0, \mathbf{H}_0)}{t^m} + \frac{C_i^p(\mathbf{E}_0, \mathbf{H}_0)}{t^{p+\frac{3}{2}}} \tag{2.3}$$

where the above constants satisfy

$$\left\{\begin{array}{l} C_r^m(\mathbf{E}_0, \mathbf{H}_0) \lesssim \|\mathbf{E}_0\|^2_{H^m(\mathbb{R}^3)} + \|\mathbf{H}_0\|^2_{H^m(\mathbb{R}^3)}\,, \\ C_i^p(\mathbf{E}_0, \mathbf{H}_0) \ \lesssim \big\|\mathbf{E}_0\big\|^2_{L^{1,p}(\mathbb{R}^3)} + \big\|\mathbf{H}_0\big\|^2_{L^{1,p}(\mathbb{R}^3)} \end{array}\right.$$

and $\|u\|_{L^{1,p}(\mathbb{R}^3)} := \big\|(1 + |x|^p)\, u\big\|_{L^1(\mathbb{R}^3)}$.

Remark 2 *The upper bounds in (2.3) are sharp in the sense that similar lower bounds can be obtained for well chosen initial data.*

Remark 3 *The statement of the theorem has to be slightly modified if one of the $\alpha_{e,j}$ vanishes while $\alpha_{\ell,m} = 0$ for all m (or vice versa). More precisely, with the same assumptions, one has to replace in the estimate (2.3) t^{-m} by $t^{-m/2}$.*

3 Comparison with the literature

The law (1.2) enters a more general class of non local (in time) constitutive laws of the form

$$\left\{\begin{array}{l} \mathbf{P}(x,\cdot) = \chi_e \star \mathbf{E}(x,\cdot), \\ \mathbf{M}(x,\cdot) = \chi_m \star \mathbf{H}(x,\cdot) \end{array}\right. \tag{3.1}$$

where (χ_e, χ_m) are time convolution causal kernels. The long time behaviour of solutions of (1.1, 3.1) has been investigated in many papers such as in [2] for bounded domains of propagation (using abstract results from semi-group theory). In these works, polynomial stability, i.e. time decay estimates of the type

$$\mathcal{E}(t) \lesssim (1+t)^{-p}, \quad \text{for some } p > 0, \tag{3.2}$$

is proven under direct assumptions on the kernels (χ_e, χ_m). In the case of Lorentz media, the assumptions used in [2] are satisfied only under the strong dissipativity assumption, i. e.

$$\alpha_{e,j}, \ \alpha_{m,\ell} > 0, \quad \forall\, j, \ell, \tag{3.3}$$

Another major difference is in the method of proof: we use here a more physically oriented method based on a modal/spectral approach.

4 Method of proof

One first writes the problem as a generalized Schrödinger equation of the form

$$\frac{d\mathcal{U}}{dt} + \mathrm{i}\,\mathcal{A}(\nabla)\,\mathcal{U} = 0 \tag{4.1}$$

where $\mathcal{U} := (\mathbf{E}, \mathbf{H}, \mathbf{P}_j, \partial_t \mathbf{P}_j, \mathbf{M}_\ell, \partial_t \mathbf{M}_\ell) \in \mathbb{R}^N$ with $N = 3\,(2 + 2\,N_e + 2\,N_m)$ and $\mathcal{A}(\nabla)$ is a first order differential operator in space. We apply the space Fourier transform

$$\mathcal{U}(x,t) \quad \longrightarrow \quad \mathbf{U}(\mathbf{k}, t),$$

so that $\mathbf{U}(\mathbf{k}, t)$ satisfies

$$\frac{d\mathbf{U}}{dt}(\mathbf{k}, t) + \mathrm{i}\,\mathcal{A}(\mathbf{k})\,\mathbf{U}(\mathbf{k}, t) = 0, \tag{4.2}$$

where $\{\mathcal{A}(\mathbf{k}), \mathbf{k} \in \mathbb{R}^3\}$ is a family of *non-normal* $N \times N$ matrices. We next derive a priori estimates for $\mathbf{U}(\mathbf{k}, t)$ before coming back to space domain via Plancherel's therorem.

We have developed two approaches to obtain estimates in the $\mathbf{k}$-space.

(I) Via frequency dependent Lyapunov functions.

This approach is *more direct* but *limited* to the strict dissipativity assumption (3.3).

We construct a $|\mathbf{k}|$-dependent and positive quadratic functional $\mathcal{L}_{m,p}(|\mathbf{k}|; \mathbf{U})$ such that

$$\frac{d}{dt}\,\mathcal{L}_{m,p}\big(|\mathbf{k}|; \mathbf{U}\big) + \Phi\big(|\mathbf{k}|\big)\,\mathcal{L}_{m,p}(|\mathbf{k}|; \mathbf{U}) \leq 0,$$

for some function $\Phi(r) > 0$ ($r \in \mathbb{R}_*^+$). Then combining Grönwall's lemma with a careful examination of the behaviour of $\Phi(|\mathbf{k}|)$ for small and large values of $|\mathbf{k}|$ leads to (2.3).

(II) Via spectral decomposition.

We use the spectrum $\{\omega_n(\mathbf{k})\}$ of $\mathcal{A}(\mathbf{k})$ which is made of the solutions of a *dispersion relation* that satisfy $\mathcal{I}m\ \omega_n(\mathbf{k}) < 0$. We then use the associated spectral decomposition of $\mathcal{A}(\mathbf{k})$ to represent the solution $\mathbf{U}(\mathbf{k}, t)$ of (4.2) as

$$\mathbf{U}(\mathbf{k}, t) = \sum e^{-\mathrm{i}\,\omega_n(\mathbf{k})\,t}\, P_n(\mathbf{k}, t)\,\mathbf{U}_0(\mathbf{k}), \tag{4.3}$$

where the operators $P_n(\mathbf{k}, t)$ are directly related to the spectral projectors of $\mathcal{A}(\mathbf{k})$. We then split the analysis in three parts:

(i) estimates for low (space) frequencies $|\mathbf{k}| < m$,

(ii) estimates for high frequencies $|\mathbf{k}| > M$,

(iii) estimates for mid-frequencies $m \leq |\mathbf{k}| < M$.

While step (iii) provides a uniform (in $\mathbf{k}$) exponential decay, the polynomial decay results from the estimates (i) and (ii) which rely on

(a) the asymptotic behaviour of each $\omega_n(\mathbf{k})$ for small and large values of $|\mathbf{k}|$,

(b) uniform bounds for the operators $P_n(\mathbf{k}, t)$.

For the above, we use the implicit function theorem and the holomorphic functional calculus.

References

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