Well-posedness and shape optimization for the Westervelt Robin boundary problem on domains with non-Lipschitz boundaries

Anna Rozanova-Pierrat[1,*] Adrien Dekkers[1,1], Michael Hinz[2,2], Alexander Teplyaev[3,3]

[1]MICS, Department of Mathematics, CentraleSupélec, University Paris-Saclay, Gif-sur-Yvette, France

[2]Department of Mathematics, Bielefeld University, Bielefeld, Germany

[3]Department of Mathematics, University of Connecticut, Storrs, USA

[*]Email: anna.rozanova-pierrat@centralesupelec.fr

Abstract

We obtain the global on-time well-posedness of the Robin type boundary valued problem for the Westervelt equation on a bounded domain with a non-Lipschitz boundary. The obtained weak solutions are considered in the domain of Laplacian and thus are more regular than H^1. The irregularity of the boundary does not allow the usual H^2-regularity. We consider the shape optimization problem for this ultrasound wave propagation model to minimize the system's total acoustic energy by the shape of the boundary for fixed source and initial data. For the Robin boundary, modeling the reflection, we prove the existence of an optimal shape realizing the infimum of the acoustic energy in a class of Lipschitz boundaries. Using its relaxation on the uniform class of domains, we prove the existence of an optimal shape realizing the minimum. **Keywords:** Westervelt equation; Robin boundary condition; fractals; shape optimization; Mosco convergence

1 Westervelt model and assumptions

Westervelt equation [14] is one of the models of non-linear acoustics describing the ultrasound propagation. We study it in a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ with Robin-type boundary conditions:

$$\begin{cases} \partial_t^2 u - c^2\Delta u - \nu\Delta\partial_t u = \alpha u \partial_t^2 u + \alpha(\partial_t u)^2 + f, \\ \frac{\partial}{\partial n} u + a(x)u = 0 \text{ on } \partial\Omega \times [0, T], \\ u(0) = u_0, \;\; u_t(0) = u_1 \text{ in } \Omega. \end{cases} \tag{1}$$

Here the coefficients c, ν, α are positive constants (corresponding respectively to the sound speed in the air, the viscosity, and non-linear effects coefficient), and the reflection coefficient $a(x) > 0$ is a continuous in $\overline{\Omega}$ function. We don't suppose any particular regularity on Ω and its boundary. Precisely, we assume that Ω is a Sobolev extension domain and its boundary $\partial\Omega$ is the support of a positive Borel measure μ, supposed d-upper regular with a real $d \in [n-1, n)$. It means that there exists a constant $c_d > 0$ such that

$$\mu(B_r(x)) \le c_d r^d, \quad x \in \partial\Omega, \quad 0 < r \le 1, \tag{2}$$

where $B_r(x)$ is the Eucludien ball centered in x of radius r.

This condition allows to have boundaries with the Hausdorff dimension $\dim_H \partial\Omega \ge d$, *i.e.* d-sets, fractals, multi-fractals, regular boundaries. The Sobolev extension property does not allow the boundary to have infinitely small collapsing parts, as fractal trees do.

2 Weak well-posedness

In the regular case of C^2-boundary, there are different well-posedness results [8, 9, 12]. Here, in the absence of the usual $H^2(\Omega)$-regularity for the elliptic problems, we propose an alternative way to solve it following [1, 3, 4], using the domain of Laplacian. One way of the proof is based on a variant of a fixed point theorem [13], once we have the well-posedness of the linear problem by the Galerkin method. Another way is to use the Mosco convergence of the variational formulations on the regular domains to an irregular one to obtain the existence of a weak solution. To prove the unicity, it is sufficient to apply an L^2-stability estimate (see (8) from [2, Theorem 1.1]). The main difficulty is the absence of a part of the Dirichlet boundary, ensuring the Poincaré inequality and uniform, on boundary geometry, control estimates of the solution. Thanks to its generalization [6], we handle this problem.

3 Shape optimization: existence

The interest in modeling ultrasound propagation comes from ultrasound imaging (HIFU), lithotripsy, or thermotherapy. Recently [10] in the lithotripsy framework was considered a shape optimization problem for the Westervelt equation considering the shape derivative on a regular boundary. The existence of such optimal shape was open, and we solved it. We define the admissible class of shapes in which we are searching for the minimum of our energy functional presenting an equivalent H^1-norm. This problem as in [10] can also be viewed in the sense of minimization of the difference between the solution u and the reference state g by $H^1([0,T]\times\Omega)$ norm since $w = u - g$ is a solution of the Westervelt equation with just a different source term and the homogeneous initial data. We follow [5, 7, 11] to prove the existence of an optimal shape realizing the infimum of the acoustic energy in a class of Lipschitz boundaries and the minimum on a compact class of uniform domains.

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