# On the construction of Shape Functions for Spacetime Trefftz-DG Formulations of Wave Problems with Perfectly Matched Layers

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### Abstract

The Trefftz method is based on the construction of shape functions which are elementwise solutions of the equation to solve. We are interested here in the construction of shape functions that allow to solve Perfectly Matched Layers (PML) formulations of the acoustic wave equation in Trefftz-DG spaces. Different approximation spaces are considered and assessed with numerical experiments.

*Keywords:* Trefftz methods, Tent-Pitching, PML

## 1 Trefftz methods

Trefftz-DG approximations of wave equations in the frequency domain have shown clear potential in controlling numerical pollution (see e.g. [1–3]). They are based on spaces of shape functions which are local solutions of the equation to be solved. In this way, the associated variational problem is posed only on the edges or faces of the elements, and the computational load required for the inversion of the associated linear system is reduced. The extension of this approach to the time domain has been done by several authors (e.g. [4] and [5]) and its implementation involves integrating the equations in space and time using time and space dependent shape functions.

In our work, we consider Trefftz functions defined as elementwise solutions. They are Discontinuous Galerkin shape functions as the continuity at the interfaces of the elements is not strongly imposed. The resulting Trefftz-DG variational formulation involves integrals on the boundaries of each cell of the mesh of the domain of interest with appropriate transmission conditions between the cells.

In spacetime Trefftz-DG formulations, the solution is computed implicitly, which implies to invert a sparse but very large matrix. To overcome this difficulty, Tent-Pitching algorithms were first introduced for hyperbolic problems in spacetime domain in [6]. They consist in building a causal mesh, which respects the wave propagation speed (see figure 1). This leads to solving the problem elementwise, which converts the original implicit scheme to a locally-implicit one. In practice, only small local matrices are inverted; either reference matrices that apply to all of the tents (structured meshes) or one matrix for each tent (unstructured meshes). These algorithms also have the major advantage of being conducive to parallel computing, which is mandatory when handling three dimensions and/or larger domains.

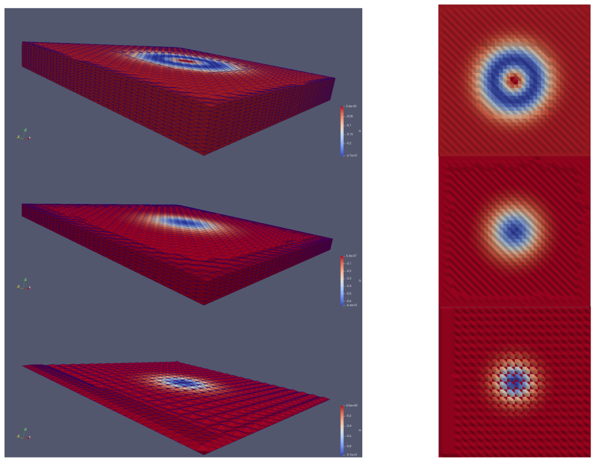

Figure 1: Evolution of Tent-Pitching meshes in 2d+time

## 2 Acoustic wave equation with PML

Perfectly Matched Layers were first introduced by Bérenger in 1994 to absorb waves and avoid spurious reflections. For instance, the acoustic wave problem with PML in the y-direction in the domain $\Omega$ can be written as :

$$\begin{cases} \rho\dfrac{\partial u_x}{\partial t} + \dfrac{\partial p}{\partial x} = 0\,, \\ \rho\dfrac{\partial u_y}{\partial t} + (\dfrac{\partial}{\partial t} + \sigma)^{-1}\dfrac{\partial}{\partial t}\dfrac{\partial p}{\partial y} = 0\,, \\ \dfrac{1}{c^2\rho}\dfrac{\partial p}{\partial t} + \dfrac{\partial u_x}{\partial x} + (\dfrac{\partial}{\partial t} + \sigma)^{-1}\dfrac{\partial}{\partial t}\dfrac{\partial u_y}{\partial y} = 0\,. \end{cases}$$

with

| | |
|---|---|
| $\Omega = \mathcal{D} \times [0, T]$, | $T$, the final time, |
| $\mathcal{D}$, the space domain, | $c$, wave speed, |
| $p(x, y, t)$, pressure, | $\rho$, density, |
| $u(x, y, t)$, velocity, | $u = (u_x, u_y)$ |

and $\sigma$ the absorption coefficient, depending only on $y$.

In classical Trefftz-DG formulations, both shape and test functions are elementwise solutions to the problem under consideration. As a consequence, the resulting variational formulation only involves surface integrals which contributes to reduce the computational cost. In the PML, it turns out that shape and test functions have to be different to keep this property. This point will be discussed during the talk.

## 3 Shape functions

The implementation of Trefftz methods requires the construction of shape functions which are solutions in each element of the problem to solve. Ideally, we would like to have polynomial basis functions, but it turns out that their construction is not that obvious when considering PML.

However, according to [8], the Green's function $G_p$ associated with the pressure can be computed analytically and so does $G_p^{\mathrm{pml}}$, the Green's function inside of the absorbing layer.

With this in mind, exact solutions of the acoustic wave equation with PML can be computed as Green's functions which are denoted $G_p^{\mathrm{pml}}$, $G_{u_x}^{\mathrm{pml}}$, $G_{u_y}^{\mathrm{pml}}$, with $u_x$ and $u_y$ referring to the velocity in the $x$- and $y$- direction. They are null when $\mathbf{t} < \frac{r}{c}$, with $r = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$ and when $\mathbf{t} > \frac{r}{c}$, they can be written as :

$$G_p^{\mathrm{pml}}(\mathbf{x}, \mathbf{y}, \mathbf{t}) = e^{-A(\mathbf{x},\mathbf{y},\mathbf{t})} cos[B(\mathbf{x}, \mathbf{y}, \mathbf{t})] G_p$$

$$G_{u_x}^{\mathrm{pml}}(\mathbf{x}, \mathbf{y}, \mathbf{t}) = \frac{\mathbf{t}\mathbf{x}}{\rho r^2} G_p^{\mathrm{pml}} - \frac{e^{-A(\mathbf{x},\mathbf{y},\mathbf{t})} G_p}{\rho r^2} \times \left( \mathbf{y} \sqrt{\mathbf{t}^2 - \frac{r^2}{c^2}} sin[B(\mathbf{x}, \mathbf{y}, \mathbf{t})] \right)$$

$$G_{u_y}^{\mathrm{pml}}(\mathbf{x}, \mathbf{y}, \mathbf{t}) = \frac{\mathbf{t}\mathbf{y}}{\rho r^2} G_p^{\mathrm{pml}} - \frac{e^{-A(\mathbf{x},\mathbf{y},\mathbf{t})} G_p}{\rho r^2} \times \left( \mathbf{x} \sqrt{\mathbf{t}^2 - \frac{r^2}{c^2}} sin[B(\mathbf{x}, \mathbf{y}, \mathbf{t})] \right)$$

With $\mathbf{x} = \{x - x_i\}_{i=1, n_s}$, $x_i$ being the source points of the Green's functions. We can achieve convergence by increasing $n_s$.

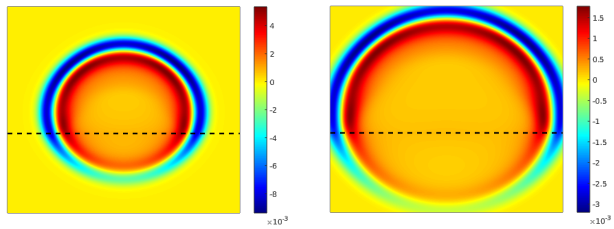

Figure 2: Pressure of the acoustic wave with the absorption coefficient $\sigma = 7$, on the left at $t = 0.31$s and on the right at $t = 0.47$s.

$$\text{and} \quad \begin{cases} A(\mathbf{x}, \mathbf{y}, \mathbf{t}) = \Sigma(\mathbf{y}) \frac{\mathbf{t}}{r^2} |\mathbf{y}| \\ B(\mathbf{x}, \mathbf{y}, \mathbf{t}) = \frac{\Sigma(\mathbf{y})}{r^2} \mathbf{x} \sqrt{\mathbf{t}^2 - \frac{r^2}{c^2}} \\ \Sigma(\mathbf{y}) = \int_0^{\mathbf{y}} \sigma(\mathbf{y}) d\mathbf{y} \end{cases}$$

Figure 2 illustrates the simulation of the acoustic wave propagation using Trefftz-DG methods in an homogeneous domain with a PML at the bottom (below the dashed black line). Here, the previously computed Green's functions are used as basis functions. We can see that the absorption is similar to PML in classical methods.

Hence, we have derived a Trefftz-DG-PML framework with exact solutions as basis functions. The implementation of Green's functions is tedious and an easier alternative would be to use polynomials. However, the computation of polynomial solutions to the PML equation is not an easy task, as will be explained during the talk. To overcome this issue, the idea of using approximate solutions as basis functions has been explored in [7] for very heterogeneous domains and would be an interesting axis to explore in the PML case.

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