Homogenization for transient waves in 1D periodic media: dispersion, interfaces and point sources

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Abstract

In the context of linear waves in 1D microstructured media, an homogenized model is proposed. It combines second-order asymptotic homogenization to account for dispersion, a reformulation into an hyperbolic system, and interface correctors to model transmssion from or towards homogeneous media. The well posedness of this "total model" is proven and its efficiency is illustrated via numerical simulations. An extension to Dirac source terms is proposed.

Keywords: homogenization, dispersion, interfaces

1 Introduction

We first focus on waves propagating in unbounded one-dimensional media ($x \in \mathbb{R}$) characterized by density $\rho_\ell(x) = \rho(x/\ell)$ and Young's modulus $E_\ell(x) = E(x/\ell)$, in terms of 1-periodic functions (ρ, E) and the periodicity length ℓ. The material displacement is denoted $u_\ell(x,t)$, and $v_\ell = \partial_t u_\ell$, $\sigma_\ell = E_\ell \partial_x u_\ell$ are the velocity field and stress field, satisfying the system:

$$\begin{cases} \partial_t v_\ell - \dfrac{1}{\rho_\ell}\partial_x \sigma_\ell = 0, \\ \partial_t \sigma_\ell - E_\ell\, \partial_x v_\ell = 0. \end{cases} \tag{1}$$

To avoid dealing with the oscillating coefficients (ρ_ℓ, E_ℓ), an homogenization process is deployed. At leading-order, it leads to a similar system with constant coefficients ρ_0 and E_0, which are respectively the mean and harmonic mean of ρ and E. The solutions of this system are reasonable approximations of (v_ℓ, σ_ℓ) for very large wavelengths λ, *i.e.* when the ratio $\varepsilon = \ell/\lambda$ vanishes.

For larger wavelengths, however, the microstructural effects must be accounted for, notably *dispersion* and *interface (or boundary) layers*, see [1] and the references therein. This can be done by increasing the order of the homogenization process in ε.

2 Stress-gradient system

Pushing the homogenization up to second order provides a family of enriched dispersive wave equations, featuring fourth-order space, time and "mixed" derivatives. In particular, the so-called (mt) model studied in [1] can be reformulated into a system inspired by stress-gradient phenomenological models (see [2]):

$$\begin{cases} \partial_t w - \dfrac{a}{\rho_0}\partial_x \sigma & = -\dfrac{a-1}{\rho_0} r, \\ \partial_t \sigma - E_0 \partial_x w & = 0, \\ \partial_t \varphi - \dfrac{a-1}{\rho_0}\partial_x \sigma & = -\dfrac{a-1}{\rho_0} r, \\ \partial_t r & = \dfrac{E_0}{\ell^2 \beta}\varphi, \end{cases} \tag{2}$$

where (w, σ) are macroscopic fields gathering the slow variations of (v_ℓ, σ_ℓ), and (φ, r) are auxiliary fields. The second-order coefficient $\beta > 0$ is imposed by the homogenization process, and the parameter a must satisfy:

$$a = -\beta_m/\beta_t \text{ with } \beta_m + \beta_t = -\beta. \tag{3}$$

The velocity v_ℓ is finally approximated as:

$$v_\ell(x,t) \approx \sum_{j=0}^{2} \ell^j P_j\left(\frac{x}{\ell}\right) \partial_x^j w(x,t) - \varphi(x,t), \tag{4}$$

where $P_0 = 1$ and the cell functions $\{P_j\}_{j=1,2}$ solve auxiliary cell problems depending on (ρ, E). A similar expansion is found for σ_ℓ, with specific cell functions Q_j. These cell functions are also used to define β.

From hyperbolic systems properties [3], we prove the following proposition:

Proposition 1 *If the parameters (β_m, β_t) satisfying (3) are chosen so that $\beta_m < 0$ and $\beta_t > 0$ (so that $a > 1$), then the system (2) is hyperbolic, the null solution is stable, and there is an associated positive conserved volume energy.*

3 Interfaces: first-order correctors

We consider now an interface at $x = 0$ between a homogeneous medium ($x < 0$) caracterized by (ρ_-, E_-) and a microstructured medium ($x > 0$) characterized by (ρ_ℓ, E_ℓ), see Figure 1. The interface is perfect, *i.e.* the velocity and stress continuity are imposed on (v_ℓ, σ_ℓ). To use the homogenized system (2) in $\mathbb{R}^+$, interface correctors must then be found. Following [1], the continuity conditions above are applied to homogenized approximations such as (4). After some reformulations, and stopping the process at first order, the following "spring-mass" transmission conditions on the macroscopic fields (w, σ) are obtained:

$$\begin{cases} [\![w]\!]_d = \ell\, A_1\, \partial_t \langle \sigma \rangle_d, \\ [\![\sigma]\!]_d = \ell\, B_1\, \partial_t \langle w \rangle_d, \end{cases} \tag{5}$$

where $[\![f]\!]_d$ and $\langle f \rangle_d$ denote the jump and mean of a function f accross an *enlarged interface* $I^d = [-d\ell, d\ell]$ introduced to ensure stability, following earlier works on interface homogenization [4]. The coefficients (A_1, B_1) above are then given by:

$$\begin{aligned} A_1 &= dE_-^{-1} + (d - P_1(0))E_0^{-1}, \\ B_1 &= d\rho_- + (d - Q_1(0))\rho_0. \end{aligned} \tag{6}$$

To ensure the positivity of these coefficients, the interface parameter d is chosen as:

$$d_{\text{opt}} := \max\left(\frac{E_- P_1(0)}{E_- + E_0}, \frac{\rho_0 Q_1(0)}{\rho_- + \rho_0}, 0 \right). \tag{7}$$

We finally prove the proposition:

Proposition 2 *The "total model" made of the system* (2) *and the jump conditions* (5) *across an enlarged interface, where the interface parameter d is given by* (7)*, is stable: a positive total energy exists and is conserved.*

As an example, Figure 1 presents a transmitted wave from an homogeneous to a laminated material for non-negligeable values of ε. The dispersive effects are clearly observed after a short propagation time, and also affect the reflected wave. All these features are well-captured by the homogenized model.

Finally, the same tools will be used to present extensions to the treatment of *point-sources* (modeled by Dirac source terms) in the microstructure.

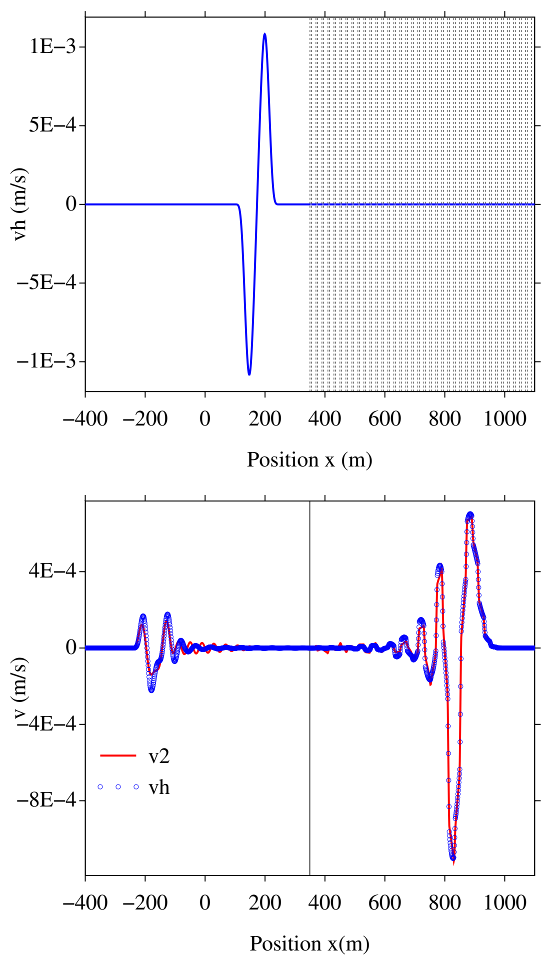

Figure 1: Transmission between homogeneous and laminated media ($\ell = 20$m). Top: initial condition, whose spectrum corresponds to $\varepsilon = \ell/\lambda \in [0, 0.4]$. Bottom: numerical reference velocity v_h (blue dots) after the transmission, and homogenized approximation including correctors in $\mathbb{R}^+$ as given by (4) (red plain line).

References

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