Maxwell's equations in presence of a tip of material with negative permittivity

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Abstract

This work is devoted to the analysis of time-harmonic Maxwell's equations in presence of a conical tip of a material with negative dielectric permittivity ε and/or negative magnetic permeability μ. When these constants ε and μ belong to some critical range, the electromagnetic field exhibits strongly oscillating singularities, such that Maxwell's equations are not well-posed in the classical $\mathbf{L}^2$ framework. Following what has been done for the 2D scalar case [1], we show how to provide an appropriate functional setting, adding to weighted Sobolev spaces the so-called outgoing propagating singularities.

Keywords: time-harmonic Maxwell's equations, metamaterial, singularities, Kondratiev weighted Sobolev spaces, T-coercivity, compact embeddings, vector potentials

1 Setting of the problem

Let Ω be a bounded domain of $\mathbb{R}^3$ which contains an inclusion $\mathcal{M}$ of a particular material (metal at optical frequency, negative index metamaterial). We assume that $\partial\mathcal{M}$ is of class C^2 except at the origin O where $\mathcal{M}$ coincides locally with a conical tip. For simplicity, we suppose that only ε has a sign-change ($\mu = 1$ everywhere): ε takes the constant value $\varepsilon_- < 0$ (resp. $\varepsilon_+ > 0$) in $\mathcal{M}$ (resp. $(\Omega \setminus \overline{\mathcal{M}})$).

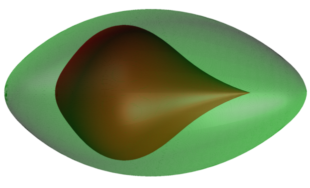

Figure 1: The geometry.

We consider the Maxwell's problem

$$\left|\begin{array}{l} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} - \omega^2\varepsilon(\boldsymbol{x})\boldsymbol{E} = i\omega\boldsymbol{J} \quad (\Omega) \\ \boldsymbol{E}\times\nu = 0 \quad (\partial\Omega) \end{array}\right. \tag{1}$$

where the current density $\boldsymbol{J} \in \mathbf{L}^2(\Omega)$ satisfies $\operatorname{div}\boldsymbol{J} = 0$. Let us introduce the scalar operator $A_\varepsilon : \mathrm{H}^1_0(\Omega) \to (\mathrm{H}^1_0(\Omega))^*$ defined by

$$\langle A_\varepsilon\varphi, \varphi'\rangle = \int_\Omega \varepsilon\nabla\varphi\cdot\nabla\overline{\varphi'}\,dx$$

for all $\varphi, \varphi' \in \mathrm{H}^1_0(\Omega)$. It has been proved in [2] that if A_ε is an isomorphism, then problem (1) has the following equivalent variational form which satisfies the Fredholm property:

$$\begin{array}{l} \text{Find } \boldsymbol{E}\in\mathbf{X}_N \text{ such that } \forall\boldsymbol{F}\in\mathbf{X}_N \\ \displaystyle\int_\Omega \mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\overline{\boldsymbol{F}}\,dx - \omega^2\int_\Omega \varepsilon\boldsymbol{E}\cdot\overline{\boldsymbol{F}}\,dx \\ \displaystyle = i\omega\int_\Omega \boldsymbol{J}\cdot\overline{\boldsymbol{F}}\,dx \end{array} \tag{2}$$

where $\mathbf{X}_N := \{\boldsymbol{E}\in\mathbf{H}_N, \operatorname{div}(\varepsilon\boldsymbol{E}) = 0\}$, with $\mathbf{H}_N := \{\boldsymbol{E}; \boldsymbol{E}, \mathbf{curl}\,\boldsymbol{E}\in\mathbf{L}^2(\Omega), (\boldsymbol{E}\times\nu)_{|\partial\Omega} = 0\}$.

A similar result holds, replacing $\mathbf{X}_N$ by a larger space $\tilde{\mathbf{X}}_N$ when A_ε is a non-injective Fredholm operator.

The purpose of the present work is to study Maxwell's problem when A_ε is not a Fredholm operator, which arises when $\varepsilon_-/\varepsilon_+ \in I_\varepsilon$, where I_ε (a bounded subset of $(-\infty, 0)$) is the so-called critical interval.

2 Scalar propagating singularities

For such critical contrasts, propagating singularities exist, that are of the form

$$\mathfrak{s}(x) = \chi(r)r^{-1/2+i\eta}\varphi(\omega) \text{ with } \eta\in\mathbb{R}.$$

Here $x = r\omega$ with $r = |x|$ and $\chi \in \mathscr{D}(\Omega)$ is a cutoff function equal to 1 near the origin. These singular functions satisfy $\operatorname{div}(\varepsilon\nabla\mathfrak{s}) = 0$ near the origin. Their span is a vector space $\mathcal{S}_\varepsilon$ of finite dimension $2N_\varepsilon$. Then, thanks to a limiting absorption principle, one can define the subspace of outgoing singularities $\mathcal{S}^{\mathrm{out}}_\varepsilon = \operatorname{span}\{\mathfrak{s}_j; j = 1, \cdots N_\varepsilon\}$ such that $q(\mathfrak{s}_j, \mathfrak{s}_k) = i\delta_{j,k}$ where $q(u,v) = \displaystyle\int_\Omega \operatorname{div}(\varepsilon\nabla v)\overline{u} - \operatorname{div}(\varepsilon\nabla u)\overline{v}$.

For $\beta\in\mathbb{R}$ and $m\in\mathbb{N}$, recall that the weighted Sobolev (Kondratiev) space $\mathrm{V}^m_\beta(\Omega)$ is defined as the closure of $\mathscr{D}(\overline{\Omega}\setminus\{O\})$ for the norm

$$\|\varphi\|_{\mathrm{V}^m_\beta(\Omega)} = \Big(\textstyle\sum_{|\alpha|\le m}\|r^{|\alpha|-m+\beta}\partial^\alpha_x\varphi\|^2_{\mathrm{L}^2(\Omega)}\Big)^{1/2}$$

and $\mathring{\mathrm{V}}^1_\beta(\Omega) = \{\varphi \in \mathrm{V}^1_\beta(\Omega) \,|\, \varphi_{|\partial\Omega} = 0\}$. Next for $\beta > 0$, setting $\mathring{\mathrm{V}}^{\mathrm{out}}_\beta(\Omega) = \mathring{\mathrm{V}}^1_{-\beta}(\Omega) \oplus \mathcal{S}^{\mathrm{out}}_\varepsilon$, define the operator $\mathrm{A}^{\mathrm{out}}_\varepsilon : \mathring{\mathrm{V}}^{\mathrm{out}}_\beta(\Omega) \to (\mathring{\mathrm{V}}^1_\beta(\Omega))^*$ such that for all $u = \tilde{u} + \sum_j a_j \mathfrak{s}_j \in \mathring{\mathrm{V}}^{\mathrm{out}}_\beta(\Omega)$ and $v \in \mathring{\mathrm{V}}^1_\beta(\Omega)$: $\langle \mathrm{A}^{\mathrm{out}}_\varepsilon u, v\rangle := \displaystyle⨍_\Omega \varepsilon \nabla u \cdot \nabla \overline{v}$ where we have set

$$⨍_\Omega \varepsilon \nabla \mathfrak{s}_j \cdot \nabla \overline{v} := -\int_\Omega \mathrm{div}(\varepsilon \nabla \mathfrak{s}_j)\overline{v}$$

(note that this integral coincides with the usual one for $v \in \mathring{\mathrm{V}}^1_{-\beta}(\Omega)$).

A main result established in [3] is the existence of $\beta_D > 0$ such that the operator $\mathrm{A}^{\mathrm{out}}_\varepsilon$ is Fredholm for all $\beta \in (0, \beta_D)$, and an isomorphism as soon as A_ε is injective (which is assumed in what follows).

3 A new framework for Maxwell's equations

The above results for the scalar problem lead to look for the solution of problem (1) in the space

$$\begin{aligned}\mathbf{X}^{\mathrm{out}}_N := \{&\boldsymbol{E} = \textstyle\sum_j a_j \nabla \mathfrak{s}_j + \widetilde{\boldsymbol{E}}; \widetilde{\boldsymbol{E}} \in \mathbf{H}_N,\\ &a_j \in \mathbb{C}, \mathrm{div}(\varepsilon \boldsymbol{E}) = 0\}.\end{aligned}$$

Note that $\mathbf{X}_N \subset \mathbf{X}^{\mathrm{out}}_N$. Using the fact that $\mathrm{A}^{\mathrm{out}}_\varepsilon$ is an isomorphism, one can check that a solution of

$$\begin{aligned}&\text{Find } \boldsymbol{E} \in \mathbf{X}^{\mathrm{out}}_N \text{ such that } \forall \boldsymbol{F} \in \mathbf{X}^{\mathrm{out}}_N\\ &\int_\Omega \mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,\overline{\boldsymbol{F}}\,dx - \omega^2 ⨍_\Omega \varepsilon \boldsymbol{E} \cdot \overline{\boldsymbol{F}}\,dx\\ &= i\omega \int_\Omega \boldsymbol{J} \cdot \overline{\boldsymbol{F}}\,dx\end{aligned} \tag{3}$$

is indeed a solution of (1). This result and the analysis of (3) rely on the key following regularity result. If $\boldsymbol{E} = \sum_j a_j \nabla \mathfrak{s}_j + \widetilde{\boldsymbol{E}} \in \mathbf{X}^{\mathrm{out}}_N$, then for any $\beta < \min(\beta_D, 1/2)$, $\widetilde{\boldsymbol{E}} \in \mathbf{V}^0_{-\beta}(\Omega)$. Moreover there is a constant $C > 0$ independent of $\boldsymbol{E}$ such that

$$\textstyle\sum_j |a_j| + \|\widetilde{\boldsymbol{E}}\|_{\mathbf{V}^0_{-\beta}(\Omega)} \le C \,\|\,\mathbf{curl}\,\boldsymbol{E}\|_\Omega.$$

As a consequence, $\|\,\mathbf{curl}\,\cdot\|_\Omega$ is a norm in $\mathbf{X}^{\mathrm{out}}_N$. Besides, one can prove the following compactness result: for any bounded sequence $E^{(n)} = \sum_j a^{(n)}_j \nabla \mathfrak{s}_j + \widetilde{\boldsymbol{E}}^{(n)}$ of $\mathbf{X}^{\mathrm{out}}_N$, there exists a subsequence such that $a^{(n)}_j$ converges in $\mathbb{C}$ and $\widetilde{\boldsymbol{E}}^{(n)}$ converges in $\mathbf{V}^0_{-\beta}(\Omega)$. Summing up, one can prove the

Theorem 1 *Fredholm alternative holds for problem* (3)*: if uniqueness holds, then the problem is well-posed.*

Concerning uniqueness, note that if $\boldsymbol{E}$ is a solution of (3) for $\boldsymbol{J} = 0$, then taking $\boldsymbol{F} = \boldsymbol{E}$, we get $\Im m \left(⨍_\Omega \varepsilon \boldsymbol{E} \cdot \overline{\boldsymbol{E}}\,dx\right) = \displaystyle\sum_j |a_j|^2 = 0$. This proves that any solution $\boldsymbol{E}$ of the homogeneous problem (3) belongs to the classical space $\mathbf{X}_N$. Such a solution is called a trapped mode by analogy with waveguides problems.

4 Some concluding remarks

Since $\mathbf{X}_N$ is a closed subset of $\mathbf{X}^{\mathrm{out}}_N$, we see by previous theorem that Fredholm alternative also holds for problem (2) set in the classical framework. But what is wrong with this formulation is that a solution of (2) is not, in general, a solution of Maxwell's equation (1).

If μ is also negative in the inclusion $\mathcal{M}$, we have to consider another scalar operator. Let $\mathrm{H}^1_\#(\Omega)$ be the subset of $\mathrm{H}^1(\Omega)$ of functions with zero mean value. Consider the operator $A_\mu : \mathrm{H}^1_\#(\Omega) \to (\mathrm{H}^1_\#(\Omega))^*$ defined by

$$\langle A_\mu \varphi, \varphi' \rangle = \int_\Omega \mu \nabla \varphi \cdot \nabla \overline{\varphi'}\,dx$$

for all $\varphi, \varphi' \in \mathrm{H}^1_\#(\Omega)$. If A_μ is a Fredholm operator, the previous results can be easily extended, using T-coercivity arguments. But if it is not, not only $\boldsymbol{E}$ has to be singular, but also $\mathbf{curl}\,\boldsymbol{E}$ (and therefore the magnetic field). For this case where both contrasts in ε and μ are critical, an appropriate functional framework is given in [3] in which Fredholmness is restored.

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