Exponentially Convergent Spectral Galerkin BEM for Elastic Wave Scattering of Cracks.

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Abstract

In this work, we extend the spectral Galerkin boundary integral solver for Helmholtz presented in [1] to elastic wave scattering by multiples two-dimensional cracks. We rigorously prove the exponential convergence rate of our proposed approximations for Dirichlet and Neumann problems. We show that solutions of the associated standard first-kind boundary integral formulations take the form of a singular function times an analytic one. Then, our method focuses on only approximating the analytic part by the spectral method. Several numerical experiments confirm our claims.

Keywords: Boundary Integral Equations, Elastic wave scattering, Spectral Methods

1 Introduction

We consider an impinging elastic wave $\boldsymbol{u}^{\text{inc}}$ in a homogeneous isotropic and infinite two-dimensional medium Ω containing M cracks, jointly denoted by Γ. The direct problem consists on finding the scattered wave $\boldsymbol{u}$ under the assumption that either the cracks are rigid (Dirichlet) or traction free (Neumann). The volume problems read

$$\begin{gathered}\Delta^*\boldsymbol{u} + \rho\omega^2\boldsymbol{u} = \boldsymbol{0} \quad \text{in} \quad \mathbb{R}^2 \setminus \Gamma,\\ \boldsymbol{u} = -\boldsymbol{u}^{\text{inc}}, \quad \text{or } T\boldsymbol{u} = -T\boldsymbol{u}^{\text{inc}} \quad \text{on } \Gamma,\\ \text{with Kupradze radiation condition,}\end{gathered}$$

where $\Delta^* := \mu\text{div grad} + (\lambda+\mu)\text{grad div}$, μ, λ being the Lamé parameters of the medium, ω the angular frequency. The traction operator is $T := 2\mu\partial_{\boldsymbol{n}} + \lambda\boldsymbol{n}\text{div} - \mu\boldsymbol{\tau}\text{curl}$, where $\boldsymbol{n}$ is the normal vector, $\boldsymbol{\tau} = (-n_2, n_1)$, and $\text{curl}\boldsymbol{u} = \partial_1 u_2 - \partial_2 u_1$. We refer to [2] for the exact form of the radiation condition. Remark that this problem is relevant to many engineering applications such as non-destructive testing of materials or fractures characterization.

These volume problems are reduced to systems of first-kind boundary integral equations (BIEs) via appropriate indirect representations by means of layer potentials. These can be solved, for instance, by local polynomials defined on appropriate meshes of the arcs Γ. Special care needs to be given to how the mesh is built as solutions may exhibit singular behaviors near the endpoints of each arc. Typically, this entails particular refinements near the endpoints are needed to recover the traditional algebraic convergence rate of local (low-order) approximation methods or super-algebraic convergence of the *hp*-methods. However, for large numbers of cracks and in the context of inverse problems or uncertain quantification (UQ), the number of degrees of freedom required by this technique becomes impractical.

Following [1], we avoid any meshing by defining global basis constructed as polynomials multiplied by a special singular function that captures the singular behavior of the real solution. By examining the polynomial expansion of the BIEs solutions, we can show that our error convergence rate is exponential in the polynomial degree, assuming that the incident wave and the geometry of each arc are given by analytic functions. This proof differs from traditional results for open arcs as we do not make use of the Mellin transform with localization of the singularities. Indeed, these proofs rely on the approximation of smooth window functions which are $\mathcal{C}^\infty$ but not analytic, thus preventing the obtention of exponential convergence rate.

Furthermore, and in contrast to [1], here we also extend the analysis to hyper-singular BIEs. The corresponding analysis was obtained using an adequate Maue's representation formula (see [3]).

2 Boundary Integral Formulation and Discretization

For Dirichlet boundary conditions, the associated BIE is

$$\sum_{j=1}^{M} \int_{\Gamma_j} \boldsymbol{E}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\phi}_j(\boldsymbol{y})ds_{\boldsymbol{by}}, = -\boldsymbol{u}^{\text{inc}}|_{\Gamma_i},$$

for every i in $\{1,\dots,M\}$, and where $\boldsymbol{E}$ denotes the standard fundamental solution of the the Navier equation [2] and $\boldsymbol{\phi}_j$ are the unknown densities defined in Γ_j for each $j \in \{1,\dots,M\}$. For the Neumann problem the corresponding formulation is

$$\sum_{j=1}^{M} T_{\Gamma_i} \int_{\Gamma_j} T_{\boldsymbol{y}}\boldsymbol{E}(\boldsymbol{x},\boldsymbol{y})^{\top}\boldsymbol{\psi}_j(\boldsymbol{y})ds_{\boldsymbol{by}}, = -T_{\Gamma_i}\boldsymbol{u}^{\text{inc}},$$

where T_{Γ_i} is the traction operator on Γ_i, and $\boldsymbol{\psi}_j$ are unknown densities defined in the corresponding arc Γ_j. The solution of the associated volume problem is then obtained by taking the action of the appropriate layered potential over the densities $\boldsymbol{\phi}$, or $\boldsymbol{\psi}$ depending on the boundary condition.

Assuming that Γ_i is the image of an analytic function $\boldsymbol{r}_i : [-1,1] \to \mathbb{R}^2$. We define $\boldsymbol{\phi}^N \circ \boldsymbol{r}_i(t) = \sum_{p=1}^{2}\sum_{n=0}^{N} a_{n,p}^i T_{n,p}(t)$, and $\boldsymbol{\psi}^N \circ \boldsymbol{r}_i(t) = \sum_{n=0}^{N} b_{n,p}^i U_{n,p}(t)$, where

$$T_{n,p}(t) = (1-t^2)^{\frac{-1}{2}}\|\boldsymbol{r}_i'(t)\|^{-1}T_n(t)\boldsymbol{e}_p,$$
$$U_{n,p}(t) = (1-t^2)^{\frac{1}{2}}U_n(t)\boldsymbol{e}_p,$$

with $\boldsymbol{e}_p$ denoting the pth canonical vector, and T_n and U_n are the nth Chebyshev polynomials of first and second kind, respectively. Then, we bound the convergence rates of $\boldsymbol{\phi}^N$ and $\boldsymbol{\psi}^N$ to $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$, respectively, as a function of N.

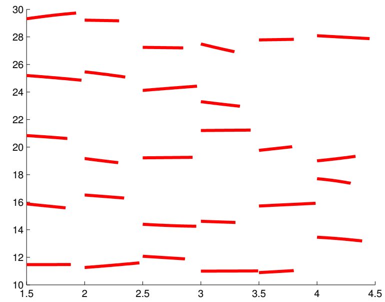
Theorem 1 *Assuming that $\boldsymbol{u}^{\text{inc}} \circ \boldsymbol{r}_i$, and $\boldsymbol{r}_i$ have analytic extension to an open region of the complex plane containing $[-1,1]$, there exist $\rho > 1$, and $\varrho > 1$, such that*

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}^N\|_{\widetilde{\mathbb{H}}^{-1/2}(\Gamma)} \leq C\rho^{-N},$$
$$\|\boldsymbol{\psi} - \boldsymbol{\psi}^N\|_{\widetilde{\mathbb{H}}^{1/2}(\Gamma)} \leq C\varrho^{-N},$$

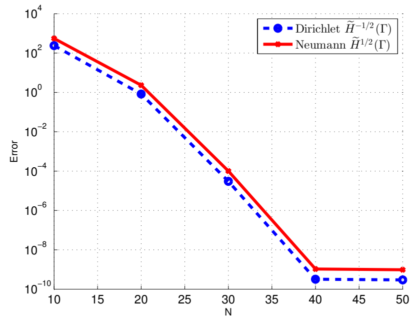
where $C > 0$ is a generic constant independent of N.

3 Numerical Results

We consider a test case with 28 arcs, see Figure 1a, and the parameters $\omega = 50$, $\lambda = 2$, $\mu = 1$. Figure 1b illustrate the errors of the method as a function of the polynomial degree.



(a) Example Geometry



(b) Error respect an overkill solution computed with $N = 100$. The incident wave is given by $\boldsymbol{u}^{\text{inc}}(\boldsymbol{x}) = \boldsymbol{d}e^{k_p \boldsymbol{x}\cdot\boldsymbol{d}}$, where $\boldsymbol{d}$ is an unitary vector with angle $\frac{\pi}{4}$ with respect to the x-axis, and $k_p = \frac{\omega}{\sqrt{\lambda+2\mu}}$.

4 Conclusions and Future Work

We have developed a novel strategy to study the solution of first-kind boundary integral equations for open arcs and applied it to the analysis of a spectral Galerkin method, for which we proved the exponential convergence rate. Current work is on to prove the convergence rates for the Nyström discretization and the application of the method in UQ problems.

References

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