Radial complex scaling/PML for anisotropic scalar resonance problems

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Abstract

We consider radial complex scaling/perfectly matched layer (PML) methods for scalar resonance problems. First we focus on isotropic problems and prove the convergence of approximations. In particular, the analysis covers the simultaneous approximation due to domain truncation and discretization, and a broad range of scaling profile functions. In addition, we obtain convergence of eigenfunctions, convergence rates and correct algebraic multiplicities of eigenvalues. Core ingredients of the analysis are the framework of T-compatible approximations of weakly T-coercive operators, and the interpretation of the domain truncation as Galerkin approximation.

In a second part we show how to extend the former results to anisotropic materials, whereat some restrictions on the choice of parameters have to be respected. To this end it is necessary to take a close look at the complex transformation of the fundamental solution, and to obtain an estimate on the numerical range of certain nonhermitian matrices.

Keywords: complex scaling, perfectly matched layer, resonances, eigenvalues, anisotropic materials, convergence analysis

1 The resonance problem

Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain such that Ω^c is nonempty and bounded. Let $B_r := \{x \in \mathbb{R}^3 : |x| < r\}$ and $r_0 > 0$ be such that $\Omega^c \subset B_{r_0}$. We consider the resonance problem to find nontrivial solutions (ω, u) to

$$\begin{aligned} -\Delta - \omega^2 u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \\ u(x) &= \frac{\mathrm{i}\omega}{4\pi} \int_{\partial B_{r_0}} u(y) \nabla_y G_\omega(x,y) \cdot \nu(y) \\ &\quad - G_\omega(x,y) \nabla_y u(y) \cdot \nu(y) \,\mathrm{d}y, \quad x \in B_{r_0}^c \end{aligned}$$

whereat $G_\omega(x,y) := h_0^{(1)}(\omega|x-y|) = \frac{e^{\mathrm{i}\omega|x-y|}}{\mathrm{i}\omega|x-y|}$ is the fundamental solution of the Helmholtz equation.

2 Radial complex scaling

Let $r_1 > r_0$, $\gamma \in \{z \in \mathbb{C} : \Re(z) \geq 0, \Im(z) > 0\}$ and $\tilde{\alpha}$ be such that $\tilde{\alpha}(r) = 0$ for $r \leq r_1$, $\tilde{\alpha}$ is continuous, $\tilde{\alpha}(r) > 0$ for $r > r_1$, $\tilde{\alpha}$ is nondecreasing, $\tilde{\alpha}$ is twice continuously differentiable in $(r_1, +\infty)$ with continuous extensions of $\partial_r \tilde{\alpha}$ and $\partial_r \partial_r \tilde{\alpha}$ to $[r_1, +\infty)$, and $\tilde{\alpha}$ and α are bounded. We define the auxiliary functions

$$\begin{aligned} \tilde{d}(r) &:= 1 + \gamma\tilde{\alpha}(r), & \tilde{r}(r) &:= \tilde{d}(r) r, \\ \alpha(r) &:= r\partial_r\tilde{\alpha}(r) + \tilde{\alpha}(r), & d(r) &:= 1 + \gamma\alpha(r), \end{aligned}$$

and $d_0 := \lim_{r \to +\infty} (\tilde{d}(r)/|\tilde{d}(r)|)$. Then for any resonance function we define formulary in spherical coordinates $\tilde{u}(x) := u(\tilde{r}(r)\hat{x})$. This definition can indeed be justified rigorously, and if $\Re(\mathrm{i}\omega d_0) < 0$, then $\tilde{u} \in H_0^1(\Omega)$ solves $a_\Omega(\omega; \tilde{u}, u') = 0$ for all $u' \in H_0^1(\Omega)$, whereat $\mathrm{P_x}(x) := |x|^{-2} x x^\top$ and

$$\begin{aligned} a_D(\omega; u, u') &:= -\omega^2 \langle \tilde{d}^2 d u, u' \rangle_{L^2(D)} \\ &\quad + \langle (\tilde{d}^2 d^{-1} \mathrm{P_x} + d(\mathrm{I} - \mathrm{P_x})) \nabla u, \nabla u' \rangle_{L^2(D)}. \end{aligned}$$

Hence we replaced the original resonance problem with the eigenvalue problem for the bounded sesquilinearform $a_\Omega(\cdot; \cdot, \cdot)$, which admits the essential spectrum $\mathbb{C} \setminus \Lambda_{d_0}$ with $\Lambda_{d_0} := \{z \in \mathbb{C} : \Re(\mathrm{i}z d_0) \neq 0\}$.

3 Convergence analysis

The convenient way to approximate this eigenvalue problem is to choose a bounded subdomain $\Omega_n \subset \Omega$ and to discretize the eigenvalue problem for $a_{\Omega_n}(\cdot; \cdot, \cdot)$ with a finite element space $X_h(\Omega_n) \subset H_0^1(\Omega_n)$. Classically the approximations by the domain truncation and the finite element discretization are analyzed in two seperate steps. However, this way it cannot be ensured that any combination of increasing domains $\Omega_n \to \Omega$ and decreasing mesh parameters $h \to 0$ yields a converging approximation. Therefore we propose a new approach to conduct the analysis, which combines several independent ideas. Consider an arbitrary sequence $X_n := X_{h_n}(\Omega_n)$ with $\Omega_n \to \Omega$ and $h_n \to 0$.

First we follow [4] and identify $H_0^1(\Omega_n)$ with $\{u \in H_0^1(\Omega)\colon u|_{\Omega_n^c} = 0\}$, and thus $X_n \subset H_0^1(\Omega)$. This way we can interprete the domain truncation as conform Galerkin approximation. Second we employ the abstract framework [3] for the Galerkin approximation of eigenvalue problems for holomorphic operator functions. To this end for each $\omega \in \Lambda_{d_0}$ we need to find an invertible operator $T \in L(H(_0^1(\Omega))$ and a compact operator $K \in L(H(_0^1(\Omega))$ such that $T^*A + K$ is coercive, whereat A is the operator associated to $a(\omega;\cdot,\cdot)$. Let $\hat{d}(r) := \lim_{\rho\to r_1+} d(\rho)$ for $r < r_1$ and $\hat{d}(r) := d(r)$ for $r \geq r_1$. Then we achieve the goal with the multiplication operator $Tu := \hat{d}^{-1}u$ for $\arg(-\omega^2 d_0^2) \in [-\pi, 0)$ and $Tu := \hat{d}\tilde{d}^{-2}u$ for $\arg(-\omega^2 d_0^2) \in [0, \pi)$. The second main assumption of [3] is that we construct operators $T_n \in L(X_n)$ such that $\lim_{n\to\infty} \|T - T_n\|_{L(X_n, H_0^1(\Omega))} = 0$. Let Π_n be the Scott-Zhang interpolant. Then we achieve the latter assumption with $T_n := \Pi_n T|_{X_n}$ by means of a suitable adaptation of the discrete commutator property. Finally we construct an appropriate function to estimate the best approximation error by the discretization error plus the truncation error, whereat the latter decreases exponentially with the layer size.

4 Anisotropic materials

A widespread observation is that PMLs for anisotropic elastodynamics can be unstable. However, this might not be true for *radial* PMLs. As a step in this direction we generalize the former convergence results to anisotropic scalar materials under some mild additional assumptions on the parameters. The recipe from the isotropic case can be reused in the main for the anisotropic case, and we focus on the points which require some additional care. At first we note that the we require a minimal distance $r_1 > c(\varsigma) r_0$ from the scattering object to the complex scaled layer, whereat $c_1(\varsigma)$ is a constant which depends on the material ς. For scattering problems also all source terms must be supported in B_{r_0}. This assumption is necessary to guarantee that the complex scaled fundamental solution $\tilde{G}_{\omega,\varsigma}(x,y)$ is well-defined and holomorphic. Secondly we need to assume $\sup_{r\geq r_1} \arg(d(r)/\tilde{d}(r)) < c_2(\varsigma)$ with a certain constant $c_2(\varsigma) > 0$. For the simplest possible scaling $\tilde{r}(r) = (1+\gamma)(r - r_1) + r_1$, $r > r_1$ this condition can be intepreted as a restriction on the damping strength. In Fig. 1 we show the geometries and meshes for an anisotropic and two isotropic examples which admit the same resonances. In Fig. 2 we present the computed spectrum with a radial PML method. We recognize a good correspondence between the computations and the reference values. The quality of the computed resonances from the anisotropic example is intermediate between the quality of the isotropic examples.

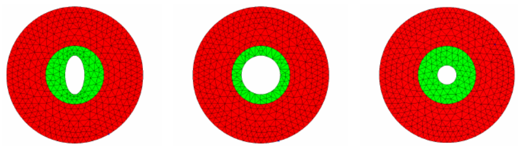

Figure 1: geometries/meshes for anisotropic and isotropic examples

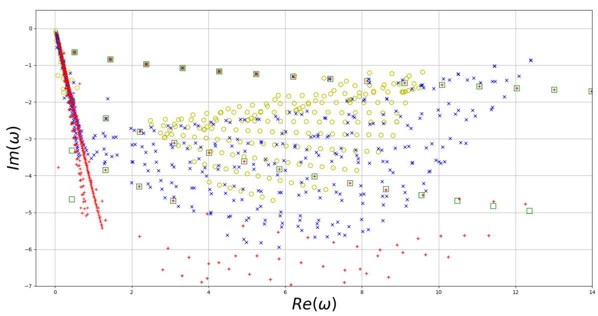

Figure 2: computed spectrum for the isotropic/scaled isotropic/anisotropic (+/o/x) case, analytic reference values (□)

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