Rational-based MOR methods for Helmholtz frequency response problems with adaptive finite element snapshots

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Abstract

We introduce several spatially adaptive model order reduction approaches tailored to parametric-in-frequency Helmholtz problems. The offline information is computed by means of adaptive finite elements, so that each snapshot lives on a grid adjusted to the considered frequency value. A rational surrogate is then assembled adopting either a least-squares or an interpolatory approach. Numerical experiments are performed to compare the proposed methodologies.

Keywords: model order reduction, rational approximation, parametric Helmholtz equation, frequency response, adaptive mesh refinement

1 Introduction

The present talk, based on [3], deals with numerical approximation of solutions to time-harmonic wave propagation problems over a range of frequencies. In particular, given any k in the interval of interest $K = [k_{min}, k_{max}]$, we look for the solution to the interior or scattering Helmholtz problem

$$\begin{cases} -\Delta u - k^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \partial_{\boldsymbol{\nu}} u = g_N & \text{on } \Gamma_N, \\ \partial_{\boldsymbol{\nu}} u - \imath k u = g_R & \text{on } \Gamma_R, \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$, $g_N \in L^2(\Gamma_N)$ and $g_R \in L^2(\Gamma_R)$, with Γ_D, Γ_N, Γ_R forming a partition of $\partial\Omega$.

Due to oscillations in the analytical solutions, accurate finite element (FE) approximations are computationally expensive and time-consuming, already for moderate frequencies. Therefore, in the multi-query context, when responses at many frequencies are of interest, their direct computation is unaffordable.

Model order reduction (MOR) methods aim at alleviating the computational cost by producing an approximation of (some functional of) the frequency response map. The produced approximation (the so-called *surrogate*) has to be close to the quantity of interest (QoI) and, at the same time, cheap to evaluate.

MOR methods rely on a two-phase procedure. The accurate computation of the offline information often requires a considerable computational effort. However, it is performed only once, and then it is stored for later use during the online phase, when the surrogate is evaluated (in real-time) at any new frequency value of interest.

2 Offline phase

The offline phase consists in two operations: (i) the sampling, namely numerical evaluation of the frequency response map for a set of frequency values (the sample points); and (ii) the surrogate assembling.

2.1 Sampling strategy

In standard MOR techniques, the snapshots are all computed on one grid of the considered physical domain. In the specific framework we are handling, this might represent a big drawback. Indeed, the analytical solution of the Helmholtz equation oscillates (the more so as the frequency increases), and it may exhibit local features or local resonance-type behavior, depending on the shape of the domain and the considered frequency values.

In constrast, the presented spatially adaptive MOR technology performs the sampling by means of the adaptive FE method. As a result, each snapshot is taken on a mesh adapted to the local features at the given parameter, and it belongs to a problem-adapted FE space.

2.2 Surrogate assembling

In [1,2] the authors have proved that the frequency-to-solution map $u\colon \mathbb{C} \to H^1_{\Gamma_D}(\Omega)$ is a meromorphic map. It is therefore sensible to look for its surrogate in the class of rational H^1-valued maps. In the present talk, we consider several techniques delivering a rational surrogate, namely the standard rational interpolation (SRI) method, which computes the rational approximant my minimizing the linearized interpolation error at the sample points, and the multipoint rational interpolation (MRI) method, which improves the SRI with the objective of reducing the number of snapshots needed to achieve a rational approximant of a certain order.

3 Discussion on h-adaptive MOR methods

The use of h-adaptive FE snapshots, each living on a different mesh of the domain, allows to save computational resources. On the other hand, it implies intrinsic difficulties: even linear combinations of snapshots cannot be easily computed. In principle, to circumvent this issue, one could express all the snapshots as elements of some common FE space. However, this calls for the construction of the so-called global mesh overlay, which entails a prohibitive computational effort and goes against the main purpose of h-adaptivity. Therefore, in all the algorithms that we propose, we never construct the global mesh overlay, but only require the evaluation of scalar products of pairs of snapshots, which is equivalent to building overlays of pairs of meshes.

4 A numerical example

Consider problem (1) with triangular domain $\Omega = \{x \in \mathbb{R}^2, 0 < x_2 < x_1 < \frac{\pi}{2}\}$, $f = 1$, $g_N = 0$, $\Gamma_D = (0, \frac{\pi}{2}) \times \{0\}$, $\Gamma_N = \partial\Omega \setminus \Gamma_D$ and $\Gamma_R = \emptyset$. Given $k^2 = 51$, we first present the computation of a snapshot using the h-adaptive FEM driven by the classical residual-based error estimator $\eta_\bullet$. In Figure 1, we show the evolution of $\eta_\bullet$ and the true error $e(z) = \|\nabla(u_\bullet(z) - u(z))\|_{L^2(\Omega)}$ as the mesh gets adaptively refined. Several peaks - caused by resonances of the discrete problem - appear before the asymptotic convergence regime is reached. To ensure accuracy, it is then crucial that the adaptive algorithm stops once the asymptotic convergence regime is achieved, namely, after all the peaks.

We now introduce the QoI $y(z) = \int_\Gamma u(z)$, with $\Gamma = \{\frac{\pi}{2}\} \times (0, \frac{\pi}{2})$ and the interval of interest $K = [1, 100]$. We construct the surrogate of the QoI by means of the SRI method, the MRI method and, for the sake of comparison, a projection-based method (POD). Building the latter two reduced models requires only 15 snapshots, as opposed to the 29 needed for SRI. We take such snapshots uniformly spaced in K. We show the results of the approximation in Figure 2. We see that the approximations yielded by the three approaches are quite similar. Moreover, we highlight that building the SRI and MRI surrogates is about 20% faster than POD.

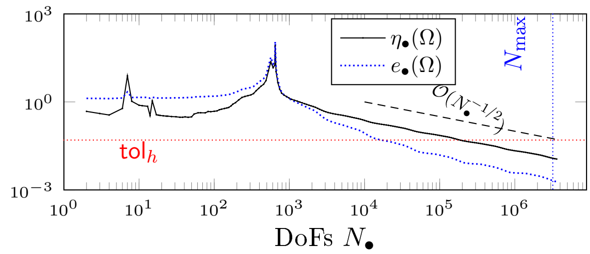

Figure 1: $\eta_\bullet(\Omega)$ and $e_\bullet(\Omega)$ vs. FEM DoFs.

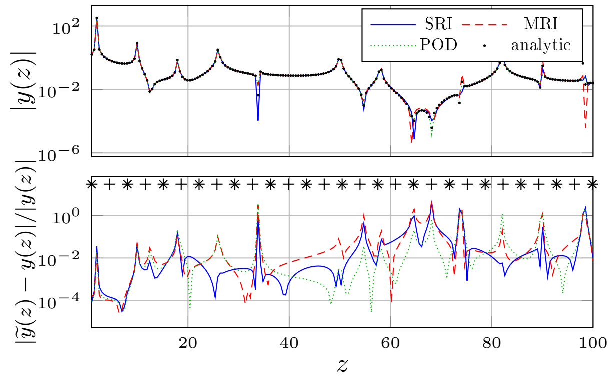

Figure 2: Comparison of the methods.

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