Efficient numerical method for time domain electromagnetic wave propagation in thin co-axial cables.

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Abstract

In this work we construct an efficient numerical method to solve 3D Maxwell's equations in coaxial cables. Our strategy is based upon an hybrid explicit-implicit time discretization combined with edge elements on prisms and numerical quadrature. One of our objectives is to validate numerically generalized Telegrapher's models that are used to simplify the 3D Maxwell equations into a 1D problem.

Keywords: Coaxial cables, Maxwell's equations, Telegrapher's models, Numerical simulation.

1 Introduction

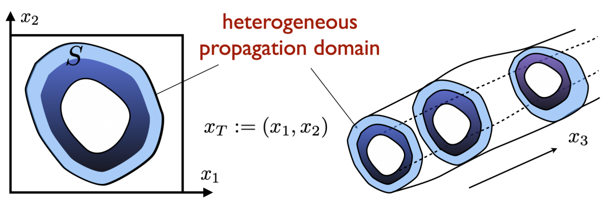

Figure 1: Coaxial cable Ω and a cross section S (non-simply connected).

Coaxial cables have transverse dimensions that are small compared to the longitudinal one : we will consider mathematically that the transverse dimension is proportional to a small parameter $\delta \ll 1$, and, using an asymptotic analysis of the 3D Maxwell equations with $\delta \to 0$, establish several 1D simplified models. Such models were rigorously derived in [1]. To our knowledge, there is no existing quantitative numerical comparison between 1D model and 3D Maxwell's theory. Doing so requires performing 3D simulations with 3D Maxwell's equations. This is a computational challenge because the cable is thin.

2 The cylindrical case

Our space-time discretization strategy, will be developed for the second order formulation of the problem, obtained after elimination of the magnetic field,

$$\varepsilon\, \partial_t^2 \mathbf{E} + \sigma\, \partial_t \mathbf{E} + \boldsymbol{\nabla} \times\, \mu^{-1}\, \boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}. \qquad (1)$$

Our approach is based on a particular rewriting of (1), that well separates the roles of the longitudinal and transverse space variables (resp, longitudinal and transverse electric fields).

In the considered applications, the wavelength λ is large compared to the diameter of the cross-section, but small compared to the size of the cable. This specificity has two impacts on the time discretization : first, an implicit scheme would be too costly given the size of the problem, second, an explicit scheme is to be avoided because the corresponding CFL condition would be too constraining. In this work we will present an efficient hybrid numerical method for solving our problem. The idea is to use an anisotropic prismatic mesh, with a transverse step size h_T and a longitudinal step size h for the space discretization (with $h_T \ll h$), a trapezoidal quadrature rule in x_3, and a hybrid implicit-explicit scheme for the time discretization [2].

To implement this method, the first step is to make a longitudinal discretization of the cable, then a transverse discretization of each section, and finally a discretization in time. The transverse field E_T will then be approximated by Nedelec elements in each section S_j and by piecewise affine elements along the longitudinal direction. On the other hand, the longitudinal field E_3 field will be approximated by $\mathbb{P}_1$ elements on each $S_{j+1/2}$ section and by $\mathbb{P}_0$ discontinuous elements along the longitudinal direction (See Figure 2).

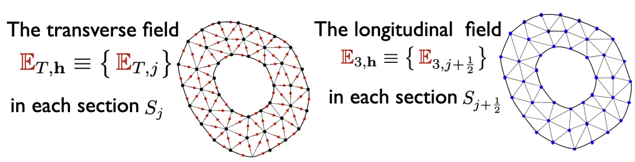

Figure 2: Two types of degrees of freedom.

In the end, the two unknown fields $\mathbb{E}_{T,j}$ and $\mathbb{E}_{3,j+\frac{1}{2}}$ alternate from one cross section to the other (See Figure 3).

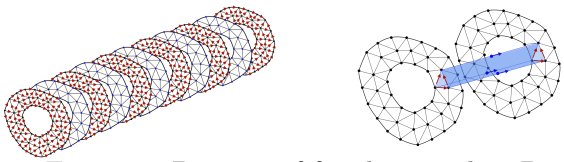

Figure 3: Degrees of freedom in the 3D.

After writing a variational formulation and using trapezoidal quadrature for x_3 integration, the algebraic form of the semi-discrete problem is,

$$\mathbb{M}_{\mathbf{h}} \frac{d^2 \mathbb{E}_{\mathbf{h}}}{dt^2} + \mathbb{K}_{\mathbf{h}} \, \mathbb{E}_{\mathbf{h}} = 0, \tag{2}$$

where $\mathbb{M}_{\mathbf{h}}$ and $\mathbb{K}_{\mathbf{h}}$ are the (infinite) mass and stiffness matrices.

$$\mathbb{M}_{\mathbf{h}} = \begin{pmatrix} \mathbf{M}_{\mathbf{h}} & 0 \\ 0 & M_{\mathbf{h}} \end{pmatrix}, \ \mathbb{K}_{\mathbf{h}} = \begin{pmatrix} \mathbf{K}_{3,\mathbf{h}} + \mathbf{K}_{T,\mathbf{h}} & \mathbf{C}_{3T,\mathbf{h}} \\ \mathbf{C}^*_{3T,\mathbf{h}} & K_{T,\mathbf{h}} \end{pmatrix}.$$

We used bold (normal) letters when they apply to transverse (longitudinal) fields. The index T means that only transverse derivatives are involved while the index 3 means that only x_3-derivatives are involved. Oppositely $\mathbf{C}_{3T,\mathbf{h}}$ couples the transverse and longitudinal fields and mixes the x_3 and transverse derivatives. Finally thanks to x_3 quadrature, we get partial mass lumping: $\mathbb{M}_{\mathbf{h}}$ is block diagonal by section.

Our method will be based on a tricky decomposition of the stiffness matrix $\mathbb{K}_{\mathbf{h}} = \mathbb{K}^i_{\mathbf{h}} + \mathbb{K}^e_{\mathbf{h}}$, where,

$$\mathbb{K}^i_{\mathbf{h}} = \begin{pmatrix} \mathbf{K}_{T,\mathbf{h}} & 0 \\ 0 & K_{T,\mathbf{h}} \end{pmatrix}, \ \ \mathbb{K}^e_{\mathbf{h}} = \begin{pmatrix} \mathbf{K}_{3,\mathbf{h}} & \mathbf{C}_{3T,\mathbf{h}} \\ \mathbf{C}^*_{3T,\mathbf{h}} & 0 \end{pmatrix}.$$

The interest of the decomposition lies in the following double observation:

- $\mathbb{K}^i_{\mathbf{h}}$ is adapted to implicit time discretization because the matrix is positive and thanks to x_3 quadrature, block diagonal by section, thus easy to invert.
- $\mathbb{K}^e_{\mathbf{h}}$ is adapted to explicit time discretization because it corresponds to the discretization of the differential operators in the x_3 direction: this matrix couples all the interfaces and has no sign.

As a consequence, we propose the following scheme,

$$\left\{ \begin{aligned} & \mathbb{M}_{\mathbf{h}} \frac{\mathbb{E}^{n+1}_{\mathbf{h}} - 2\,\mathbb{E}^n_{\mathbf{h}} + \mathbb{E}^{n-1}_{\mathbf{h}}}{\Delta t^2} + \ \mathbb{K}^e_{\mathbf{h}}\, \mathbb{E}^n_{\mathbf{h}} \\ & \qquad\qquad\qquad + \mathbb{K}^i_{\mathbf{h}} \, \{\mathbb{E}^n_{\mathbf{h}}\}_\theta = 0, \\ & \{\mathbb{E}^n_{\mathbf{h}}\}_\theta := \theta\, \mathbb{E}^{n+1}_{\mathbf{h}} + (1-2\theta)\, \mathbb{E}^n_{\mathbf{h}} + \theta\, \mathbb{E}^{n-1}_{\mathbf{h}}. \end{aligned} \right. \tag{3}$$

Theorem 1 *A sufficient stability condition for the fully discrete scheme* (3) *is that* $\theta > \frac{1}{4}$ *and,* c^+ *being the maximum celerity* c*, with* $\varepsilon\,\mu\,c^2 = 1$.

$$\frac{c^+ \Delta t}{h} < \sqrt{\frac{4\theta - 1}{4\theta}}. \tag{4}$$

3 The varying cross-section case

In the presence of deformations, the method needs to be modified. In order to preserve the longitudinal / transverse decoupling, we propose a hybrid method combining a conforming discretization in the longitudinal variable and a discontinuous Galerkin method in the transverse ones. This method is designed in order to coincide with the previous one in the cylindrical parts of the cable.

4 Numerical experiments

In Figure 4, we represent the norm $|\mathbf{E}^\delta_T|$ on the boundary $\partial\Omega$ of the reference cable, and $|\mathbf{E}^0_T|$ that is obtained by post-processing the solution of the limit model. We observe that the norm of 3D field cannot be distinguished from the one of the limit field when $\delta = 10^{-3}$.

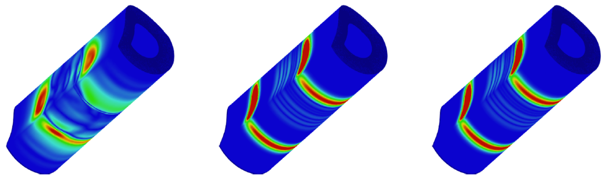

Figure 4: Left $|\mathbf{E}^\delta_T|$ with $\delta = 1$, Center $|\mathbf{E}^\delta_T|$ with $\delta = 10^{-3}$, Right $|\mathbf{E}^0_T|$ at $t = 4$.

In figure 5, we compare the evolution of the 1D (limit) voltage $V(x_3, t)$ issued from the numerical resolution of the 1D limit model, to the 1D voltage $V^\delta(x_3, t)$ for the 3D problem, and obtained by post-processing the 3D solution $\mathbf{E}^\delta_T$.

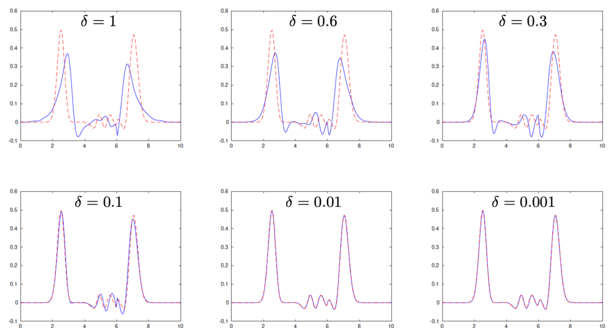

Figure 5: The voltages V^δ and V at $t = 4$.

The limit solution V is in red while V^δ is in blue. The numerics confirm that V^δ converge towards V.

References

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