Wave propagation in unbounded quasiperiodic media, Part 1: the absorbing case

Pierre Amenoagbadji[1,*], **Sonia Fliss**[1], **Patrick Joly**[1]
[1]POEMS (UMA-ENSTA Paris-IPP), Palaiseau, France
*Email: pierre.amenoagbadji@ensta-paris.fr

Abstract

This work is devoted to the numerical resolution of the Helmholtz equation in a 1D unbounded quasiperiodic absorbing medium. Using the definition of quasiperiodicity, this problem is lifted onto a 2D non-elliptic problem with periodic coefficients. The periodicity of the new problem allows to adapt some tools developed for the elliptic case [2]. However, the non-elliptic nature of the 2D PDE makes its mathematical and numerical analysis more delicate.

Keywords: quasiperiodic media, waveguides

1 Introduction

We are interested in the Helmholtz equation

$$-\big(\mu_\theta\, u'\big)' - \rho_\theta\, \omega^2\, u = f \quad \text{in} \quad \mathbb{R}, \qquad (1)$$

where μ_θ and ρ_θ have positive upper and lower bounds. The source term $f \in L^2(\mathbb{R})$ has a compact support denoted by $(-a, a)$, where $a > 0$. We additionally assume that $\mathfrak{Im}\,\omega > 0$.

Under these assumptions, (1) admits a unique solution in $H^1(\mathbb{R})$. Our objective is to *solve* (1) *numerically* when μ_θ and ρ_θ are ***quasiperiodic***, that is, when there exists $\theta \in (0, \pi/2)$ and 1–periodic coefficients μ_p, $\rho_p \in \mathscr{C}^0(\mathbb{R}^2)$ such that

$$\mu_\theta(x) = \mu_p(\vec{e}_\theta\, x) \quad \text{and} \quad \rho_\theta(x) = \rho_p(\vec{e}_\theta\, x), \quad (2)$$

where $\vec{e}_\theta = (\cos\theta, \sin\theta)$ – see Figure 1. Note that without loss of generality, μ_θ and ρ_θ could also be locally perturbed quasiperiodic functions, where the local perturbation can be supposed to be compactly supported in $(-a, a)$.

Using the properties of μ_θ, ρ_θ and f, we want to solve (1) by constructing transparent conditions of Dirichlet-to-Neumann (DtN) type:

$$\pm(\mu_\theta\, u')(\pm a) + \lambda_\theta^\pm\, u(\pm a) = 0, \qquad (3)$$

where the *DtN coefficients* $\lambda_\theta^\pm$ are computed by solving problems of the following generic form: *Find* $u_\theta \in H^1(\mathbb{R}_+)$ *such that*

$$\left|\begin{array}{l} -\big(\mu_\theta\, u_\theta'\big)' - \rho_\theta\, \omega^2\, u_\theta = 0, \quad \text{in} \quad \mathbb{R}_+^*, \\ \qquad\qquad u_\theta(0) = 1. \end{array}\right. \qquad (4)$$

The quasiperiodicity of μ_θ and ρ_θ can be exploited to solve (4). The idea to do so is to use as in [1] that the study of an elliptic quasiperiodic PDE comes down to the study of a 2D non-elliptic periodic PDE.

2 Lifting in a periodic 2D PDE

As the coefficients μ_θ and ρ_θ in (4) are defined as traces of 2D functions along the half-line $\vec{e}_\theta\,\mathbb{R}_+$, the main idea is to seek u_θ as the trace along the same line of a 2D function U_θ. Using the chain rule $\big[U_\theta(\vec{e}_\theta\, x)\big]' = (D_\theta U)(\vec{e}_\theta\, x)$ with $D_\theta := \vec{e}_\theta\cdot\nabla$, and exploiting the periodicity of μ_p and ρ_p in their first variable, it is natural to introduce the half-guide problem: $(y_1, y_2) \in \Omega := (0,1) \times \mathbb{R}_+^*$,

$$\left|\begin{array}{ll} -D_\theta\left(\mu_p\, D_\theta U_\theta\right) - \rho_p\, \omega^2\, U_\theta = 0 & (\Omega), \\ U_\theta = \varphi & (y_2 = 0), \\ U_\theta \text{ is periodic wrt. } y_1, & \end{array}\right. \qquad (5)$$

where $\varphi \in \mathscr{C}^0(\mathbb{R})$ is an arbitrary 1–periodic function that must satisfy $\varphi(0) = 1$ for the sake of consistency with $u_\theta(0) = 1$.

By Lax-Milgram's theorem, (5) admits a unique solution U_θ which belongs to

$$H^1_\theta(\Omega) := \big\{U \in L^2(\Omega),\ D_\theta U \in L^2(\Omega)\big\}.$$

Furthermore, u_θ is given by $u_\theta(x) = U_\theta(\vec{e}_\theta\, x)$.

3 Resolution of the half-guide problem

The periodicity of μ_p, ρ_p and the well-posedness of (5) allow one to show that for $\varphi \in L^2(0,1)$ and $\ell \in \mathbb{N}$, $U_\theta(\varphi)$ has the structure:

$$U_\theta(\varphi)(\cdot + \ell\vec{e}_2) = U_\theta(\mathcal{P}^\ell\varphi)(\cdot) \qquad (6)$$

where $\mathcal{P} : \varphi \mapsto U_\theta(\varphi)|_{y_2=1} \in \mathcal{L}(L^2(0,1))$ is the so-called *propagation operator*. Provided that $\mathcal{P}$ is known, $U_\theta(\varphi)$ can be obtained using the solutions of ***local cell problems*** ($\mathcal{C} := (0,1)^2$)

$$U_\theta(\varphi)(\cdot + \ell\vec{e}_2)\big|_{\mathcal{C}} = E^0(\mathcal{P}^\ell\varphi) + E^1(\mathcal{P}^{\ell+1}\varphi), \ (7)$$

where E^0 and E^1 satisfy the PDE in (5) in $\mathcal{C}$, with periodic conditions in the y_1 direction and Dirichlet conditions as explained in Figure 3.

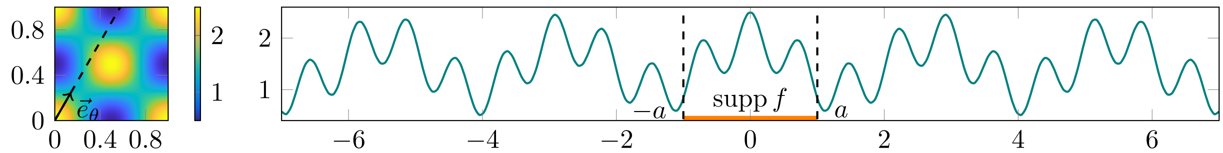

Figure 1: $\mu_p(y_1, y_2) = 1.5 + \cos(2\pi y_1)\,\cos(2\pi y_2)$ (*left*), its quasiperiodic trace along $\vec{e}_\theta$, $\theta = \pi/3$ (*right*)

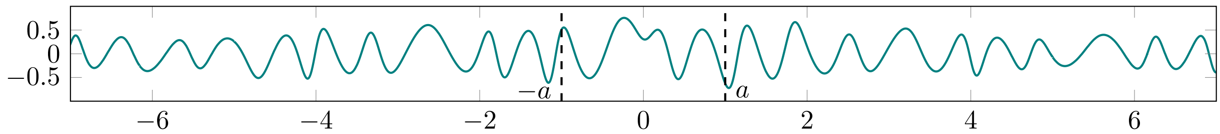

Figure 2: Solution of (1) for $\omega = 10 + 0.1\mathrm{i}$

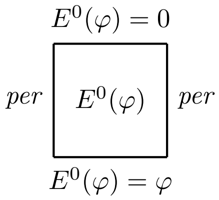

Figure 3: Local cell problems

By imposing that U_θ defined by (7) has a directional derivative which is continous accross the interface $\{y_2 = 1\}$, one deduces that $\mathcal{P}$ satisfies the ***stationary Riccati equation***

$$\mathcal{T}^{10}\,\mathcal{P}^2 + (\mathcal{T}^{00} + \mathcal{T}^{11})\,\mathcal{P} + \mathcal{T}^{01} = 0, \qquad (\mathscr{R})$$

where $\mathcal{T}^{jk}$ are *local cell DtN operators* defined from the E^j's. One shows that $\mathcal{P}$ is the unique solution of ($\mathscr{R}$) with a spectral radius $\rho(\mathcal{P}) < 1$.

Our method is very similar to [2], but its justification is more delicate due to the *non-elliptic* principal part of the operator in (5). In particular, this non elliptic nature induces a lack of compactness, and the spectral properties of the propagation operator $\mathcal{P}$ differ than the ones for the elliptic case (*cf* [2]).

4 Discretization and numerical results

Along the justification of the method, we also focus on the Finite Elements discretization of the 2D local cell problems solved by E^0 and E^1.

One natural idea to approximate E^0 and E^1 is to solve the local cell problems on arbitrary unstructured 2D meshes. Although this approach always gives efficient results, it seemed more judicious to introduce a *quasi-1D* approach. As for the method of characteristics, the main idea is to exploit the fibered structure of the operator in (5) to solve 1D bounded cell problems, and to "concatenate" the solutions to get E^0 and E^1. This allows to approximate the local DtN operators $\mathcal{T}^{jk}$ and solve the Riccati equation.

The solution U_θ can then be computed cell by cell (see Figure 4), and the DtN coefficients $\lambda_\theta^\pm$ in (3) can be deduced. Finally, one can reconstruct u in the whole line (see Figure 2).

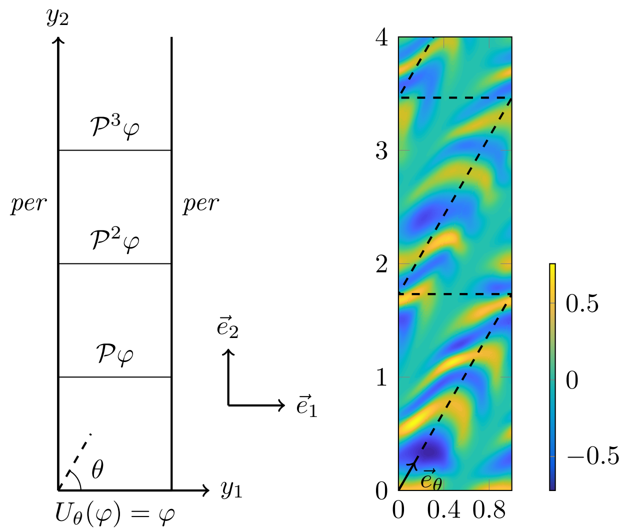

Figure 4: Half-guide and U_θ for $\omega = 10 + 0.1\mathrm{i}$

References

[1] D. Gérard-Varet, N. Masmoudi, Homogenization in polygonal domains, *Journal of the European Mathematical society* (2011)

[2] J. Patrick, J-R. Li, S. Fliss, Exact boundary conditions for periodic waveguides containing a local perturbation, *Commun. Comput. Phys* (2006)