Surface identification through back-scattering of an electromagnetic planar wave, by rational approximation

Paul Asensio[1,*], **Laurent Baratchart**[1], **Juliette Leblond**[1], **Martine Olivi**[1], **Fabien Seyfert**[1]

[1]FACTAS, Inria Sophia Antipolis-Méditerranée, Valbonne, France

*Email: paul.asensio@inria.fr

Abstract

By measuring the scattered electromagnetic field produced by a plane wave on a smooth object at various frequencies, we consider the inverse problem (nondestructive testing) of identifying the shape and certain physical characteristics of the object from the recovery of some singularities. When the object is smooth, these singularities coincide with the poles of some transfer function, that can be estimated by performing best quadratic rational or meromorphic approximation. We study the high-frequency behaviour of the scattered field and the separated contributions of its "optic" and "creeping waves" parts outside the measured frequency band and their respective influence in the reconstruction of the poles of the transfer function.

Keywords: Scattering, Inverse Problems, Shape Identification, High-frequency, Rational Approximation.

1 Introduction

We consider the scattering of a plane wave by an object Ω as described in Figure 1. In this back-scattering configuration, we measure the total electric field E on a finite band of frequency at the point $X_0 \in \mathbb{R}^3$ (outside the object) of emission of the input plane wave. The goal of this study is to identify properties of the object from the poles of the transfer function:

$$F(k) = \frac{E_{sc}(k, X_0) \cdot E_0}{E_{inc}(k, X_0) \cdot E_0},$$

where k is the spatial pulsation (proportional to the frequency), and where the incident electric field E_{inc}, which has a direction of propagation $v \in \mathbb{R}^3$, and the scattered electric field E_{sc} are given at $X \in \mathbb{R}^3$ (outside the object) by:

$$E_{inc}(k, X) = E_0 \, e^{ikv \cdot X} \, , \; E_0 \in \mathbb{R}^3 \, ,$$

$$E_{sc} = E - E_{inc} \, .$$

When the object is convex and smooth, the function F is meromorphic with poles lying in the upper half-plane. The scattered electric field is a solution to the Helmholtz equation outside Ω:

$$\Delta E_{sc} + k^2 E_{sc} = 0, \qquad (1)$$

$$\nabla \cdot E_{sc} = 0, \qquad (2)$$

$$\lim_{r\to\infty}(r(X \cdot \nabla - ik)E_{sc}) = 0 \, , \; r = |X| \, , \qquad (3)$$

$$(E_{sc} + E_{inc}) \times \nu = 0 \text{ on } \partial\Omega \, , \qquad (4)$$

for the outer normal vector ν to $\partial\Omega$.

In order to perform the rational approximation of F, we need to describe its behaviour, hence that of the field E_{sc}, at high frequency (outside the band of measured frequencies). We first consider the case in which the object Ω is a spherical PEC (Perfectly Electric Conductor) of radius a in the back-scattering orientation ($X = -rv$ with $r > a$). In this case, the solution is given by its expansion (Mie series), [2]:

$$E_{sc}(k, -rv) = \frac{E_0}{kr} \sum_{n=1}^{\infty} i^n \left(n + \frac{1}{2}\right) \times$$
$$\left[\frac{J_n(ka)H_n(kr)}{H_n(ka)} - i\frac{J_n'(ka)H_n'(kr)}{H_n'(ka)}\right],$$

where the H_n are the spherical Hankel functions of the second kind and the J_n are the spherical Bessel functions of the first kind.

As the partial sums of these series converge slowly when k increases, it is not sufficient to model the high-frequency behaviour of E_{sc}.

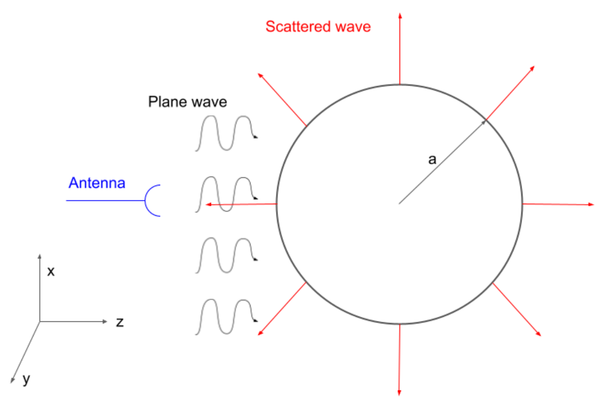

Figure 1: Setting of the scattering by a sphere.

2 Optical part

In order to study the high frequency behaviour of F, we consider a well-behaving ansatz under the form of a so-called Luneberg-Kline series, [1]. For $N > 0$:

$$E_{sc}(k, X) = \sum_{n=0}^{N} \frac{A_n(X)}{(ik)^n} e^{ikS(X)} + o\left(\frac{1}{k^N}\right). \quad (5)$$

By substituting this form into (1) to (4), we obtain the eikonal equation (6) and an infinite list of transport equations (7) that allow us to compute the ($\mathbb{R}$-valued) phase S and the sequence of ($\mathbb{R}^3$-valued) coefficients $(A_n)_{n\in\mathbb{N}}$:

$$|\nabla S|^2 = 1, \quad (6)$$

$$(\Delta S + 2\nabla S \cdot \nabla) A_n = -\Delta A_{n-1}. \quad (7)$$

using the boundary conditions that for all $Y \in \partial\Omega$:

$$\begin{aligned} A_0(Y) \times Y &= -E_0 \times Y, \\ \forall n \in \mathbb{N}^*,\ A_n(Y) \times Y &= 0, \\ \forall n \in \mathbb{N},\ \nabla S \cdot A_n(Y) &= -\nabla \cdot A_{n-1}(Y), \\ S(Y) &= v \cdot Y. \end{aligned}$$

Using these equations, we can compute the successive terms of the Luneberg-Kline series for a PEC. In the back-scattering case, the first terms are given by:

$$\begin{aligned} S &= (r - 2a), \\ A_0 &= -\frac{a}{2r - a} E_0, \\ A_1 &= -2\frac{(r-a)^2}{(2r-a)^3} E_0, \\ A_2 &= 2\frac{(r-a)(2r^2 - 4ra + 3a^2)}{(2r-a)^5} E_0. \end{aligned}$$

As we will show in section 4, this asymptotic expansion is not precise enough for our purpose and we thus consider, in the next section, terms that were neglected in the Luneberg-Kline series.

3 Creeping wave

In order to deeper understand the links between Luneberg-Kline expansions and Mie series, we apply Watson transformation [3,4] to the latter, as a way to transform the sum into an integral using the residue theorem. We show that, far from the spherical object Ω (r large):

$$\begin{aligned} E_{sc}(k, -rv) &= P(ka)\frac{\exp(ikr)}{kr} E_0, \\ P(ka) &= P_c(ka) + P_o(ka), \end{aligned}$$

where the optical part P_o has the same high-frequency behaviour as the Luneberg-Kline series (5), and the creeping wave part P_c has the following behaviour:

$$P_c(\rho) \sim \frac{\tau^4 e^{i\frac{\pi}{3}}}{\beta_1 \mathrm{Ai}(-\beta_1)^2} \exp\left(i\pi\rho - e^{-i\frac{\pi}{6}}\tau\pi\beta_1\right),$$

where $\tau = \tau(\rho) = \left(\frac{\rho}{2}\right)^{1/3}$ and β_1 is the first zero of the derivative Ai' of the Airy function Ai.

4 Numerical comparison

As P_c exponentially decreases when k goes to infinity, it seems negligible with respect to P_o. Nevertheless, for a study of the compared importance of this terms when dealing with finite frequencies, we choose a frequency of 5 GHz for a sphere of radius $a = 0.15/2$ at a distance $r = 1$. Our computations show that the three optics terms represent respectively 86.11%, 4.87% and 0.01% of the field, while the creeping wave term represents 7.55% of the field. This confirms that in order to reconstruct the poles of the transfer function, we will need to use the creeping wave part of the scattered field, even if it is negligible w.r.t. the optic part at high-frequency.

References

[1] D. Bouche, F. Molinet and R. Mittra, *Asymptotic Methods in Electromagnetics*, 1st edition, Springer Berlin Heidelberg, Berlin, 1997.

[2] R. F. Harrington, *Time-Harmonic Electromagnetic Fields*, 2nd edition, Wiley, London, 2001.

[3] T. Senior, Analytical and Numerical Studies of the Back Scattering Behavior of Spheres, *The University of Michigan Radiaton Laboratory Report* **7030-T** (1965).

[4] C. A. Valagiannopoulos, An overview of the Watson transformation presented through a simple example, *Progress In Electromagnetics Research* **75** (2007), pp. 137–152.