The WaveHoltz Heterogeneous Multiscale Method for the Helmholtz Equation

Amit Rotem[1,*], **Daniel Appelö**[1], **Olof Runborg**[2]
[1]Michigan State University, East Lansing, United States
[2]Royal Institute of Technology, Stockholm, Sweden
*Email: rotemami@msu.edu

Abstract

We are interested in finding solutions to wave equations posed in materials with rapidly varying coefficients, and time harmonic sources. For these problems, direct discretization is prohibitively costly, and instead multiscale methods are used. There are several multiscale methods, e.g. the popular heterogeneous multiscale methods (HMM) [4, 5], that directly discretize in the frequency domain. In this work we instead start in the time-domain and combine a HMM method for the wave equation [2, 3] with the newly introduced WaveHoltz method, [1]. Each WaveHoltz iteration marches the wave equation towards the time-periodic Helmholtz solution. WaveHoltz has many advantages compared to traditional Helmholtz solvers: it is positive definite, has a bounded condition number, is memory lean and can be parallelized, etc. All of these advantages will carry over to the multiscale method we present here. In addition our approach also eliminates the boundary errors present in other multiscale methods for the Helmholtz equation, where elliptic micro problems are used.

Keywords: HMM, Helmholtz, WaveHoltz

1 Introduction

Consider the Helmholtz equation on a smooth domain Ω and at frequency ω,

$$\nabla \cdot (A(x)\nabla u) + \omega^2 u = s(x), \qquad x \in \Omega.$$

The WaveHoltz method approximates the solution to this equation (for Dirichlet or Neumann boundary conditions) by iterating the equation, $v^{n+1} = \Pi v^n, v^0 \equiv 0$, where

$$\Pi v = \frac{2}{T}\int_0^T \left(\cos(\omega t) - \frac{1}{4}\right) w(x,t)\,\mathrm{d}t, \quad T = \frac{2\pi}{\omega},$$

where T is the period, and w solves

$$\begin{aligned} w_{tt} &= \nabla \cdot (A(x)\nabla w) - s(x)\cos(\omega t), \quad x \in \Omega,\\ w(x,0) &= v^n(x), \qquad w_t(x,0) \equiv 0. \end{aligned}$$

The WaveHoltz method thus finds the solution to the Helmholtz equation by repeatedly solving the wave equation [1].

The methods we propose work in multi-D but for brevity, we only consider the 1D case,

$$(A^\varepsilon(x)u_x)_x + \omega^2 u = s(x), \quad x \in [a,b]. \qquad (1)$$

The source s is independent of ε, but the coefficient $A^\varepsilon(x)$ is assumed to vary rapidly with smallest scale $\varepsilon \ll 1$, and has a scale separation property. We can for instance take locally periodic functions $A^\varepsilon(x) = A(x, x/\varepsilon)$, where A is 1-periodic in the second argument. Then the solution u can be expanded as

$$u(t,x) = u_0(t,x) + \varepsilon u_1(t,x,x/\varepsilon) + \mathcal{O}(\varepsilon^2),$$

where u_0 is independent of ε and u_1 is locally periodic in x. Here we find u_0 by applying the WaveHoltz method to the HMM discretization of the problem

$$w_{tt} = (A^\varepsilon(x)u_x)_x - s(x)\cos(\omega t), \quad x \in [a,b].$$

We use the HMM method in [2], which solves the wave equation by discretizing the domain into $N+2$ points with uniform spacing H. The solution is evolved using the second order centered differences in time and space:

$$\begin{aligned} W_j^{n+1} = 2W_j^n - W_j^{n-1} + \frac{K^2}{H}\left(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right)\\ -K^2 s(X_j)\cos(\omega t^n), \end{aligned}$$

with initial data $W_j^0 = v(X_j)$. Here

$$F_{j+\frac{1}{2}} = F\left(x_{j+\frac{1}{2}}, P_{j+\frac{1}{2}}\right), \; P_{j+\frac{1}{2}} = \frac{1}{H}\left(W_{j+1}^n - W_j^n\right)$$

The function F represents the output from a microscale solver which evolves the wave-equation on a dense grid spanning $x \in [-\eta, \eta]$ for $\eta \ll H$, and $t \in [-\tau, \tau]$ for $\tau \ll K$. This microscale solver also uses a centered difference scheme with grid spacing h and timestep k:

$$w_i^{\ell+1} = 2w_i^\ell - w_i^{\ell-1} + \frac{k^2}{h}\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right),$$

$$f_{i+\frac{1}{2}} = \frac{1}{h}\Big(A(X_{j+\frac{1}{2}} + x_{i+1})w_{i+1}^\ell - A(X_{j+\frac{1}{2}} + x_i)w_i^\ell\Big),$$

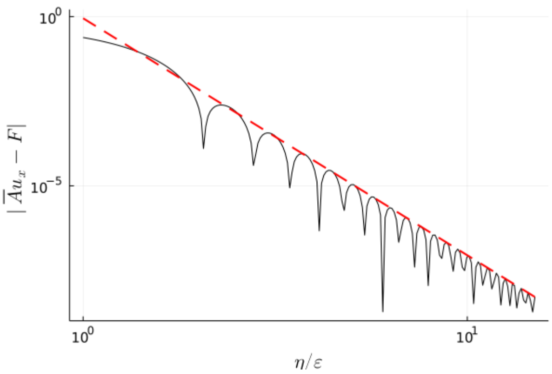

Figure 1: Convergence of the numerical flux produced by the micro-scale solver to the homogenized flux; for $p = 1, q = 5$ we see 7th order convergence (red line).

with initial data $w_i^0 = x_i P_{j+\frac{1}{2}}$. The computational domain of the micro-solver is $[-\eta', \eta']$ where $\eta' = \eta + \tau\sqrt{\max A^\varepsilon}$. The size of this patch follows from domain of dependence considerations, and guarantees that the microscale boundary conditions do not influence the solution in $[-\tau, \tau] \times [-\eta, \eta]$. Note that both η and τ are chosen to be of size $O(\varepsilon)$ which makes the computational cost of the microscale solver virtually independent of ε.

Using the micro-scale solution, the quantity $F_{j+\frac{1}{2}}$ is the average:

$$F_{j+\frac{1}{2}} \leftarrow \frac{4}{\eta\tau} \int_{-\tau}^{\tau} \int_{-\eta}^{\eta} w(x,t) \mathcal{K}(x/\eta)\, \mathcal{K}(t/\tau)\, \mathrm{d}x\, \mathrm{d}t,$$

where $\mathcal{K} = \mathcal{K}^{p,q}$ is a high-order local averaging kernel derived in [2]. Here p and q control the approximation order of the kernel.

Note that, in the micro-solver, the forcing term $s(x)\cos(\omega t)$ can be excluded as it is approximately constant over $[-\tau, \tau] \times [-\eta, \eta]$, and any constant solution will result in a net-zero contribution to the integral. A detailed convergence analysis following [1, 2] will be presented at the conference.

Numerical Results

We apply this method to problem (1) on $[-0.5, 0.5]$ with, $A^\varepsilon(x) = 1.1+\sin(2\pi x/\varepsilon)$, $s(x) = 170\omega x e^{-144x^2}$, $\omega = 15$, and $\varepsilon = 10^{-5}$, and homogeneous boundary conditions. For this problem, we have a homogenized solution which represents the $\mathcal{O}(1)$ terms of the expansion in ε; this solution is a solution to the same problem with $A \equiv \sqrt{21}/10$.

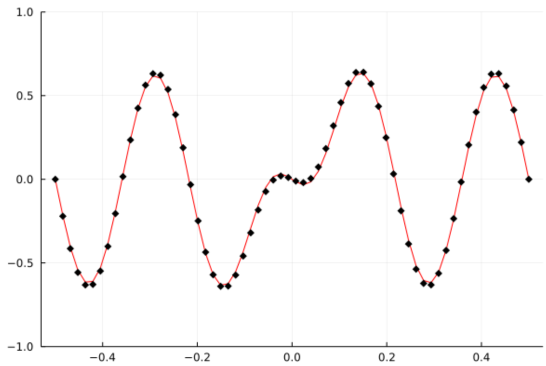

Figure 2: The red line represents the homogenized solution, and the black diamonds represent the solution found by WaveHoltz-HMM.

We first confirm the rate of convergence of the kernels in Fig. 1. Then we use the HMM solver with $N = 62$ interior points for the macro-solver, and 256 points for the micro-solver. We choose $\eta = \tau = 10\varepsilon$, and the seventh order kernel, $\mathcal{K}^{1,5}$. This solver drives the WaveHoltz method and the results are presented in Fig. 2.

References

[1] D. Appelö, F. Garcia, and O. Runborg, WaveHoltz: iterative solution of the Helmholtz equation via the wave equation, *SIAM J. Sci. Comp.* **42** (2020): A1950-A1983.

[2] B. Engquist, H. Holst, and O. Runborg, Multi-scale methods for wave propagation in heterogeneous media, *Comm. Math. Sci.* **9** (2011):33–56.

[3] A. Abdulle, and M. J. Grote, Finite element heterogeneous multiscale method for the wave equation, *Multiscale Model. Simul.*, **9**(2):766–792, 2011.

[4] P. Ciarlet Jr. and C. Stohrer, Finite element heterogeneous multiscale method for the classical Helmholtz equation, *C. R. Acad. Sci.*, **352**, 755–760, 2014.

[5] M. Ohlberger M and B. Verfürth, A new heterogeneous multiscale method for the Helmholtz equation with high contrast, *Multiscale Model. Simul.* **16**(1):385–411, 2018.