# A comparison of Hermite and Lagrange finite element methods for the wave equation

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**Abstract**

We use finite element methods with Hermite and Lagrange interpolation polynomials to solve the wave equation and compare the performance of these methods. When the same time-step is used for both we see comparable accuracy and computation time. When the time-step is chosen according to the stability properties of the methods we see that Hermite finite elements can produce a numerical solution up to twice as fast.

**Keywords:** Hermite interpolation, Lagrange interpolation, finite element method, wave equation

## 1 Hermite finite elements

We use rectangular elements in $2D$ with Langrange and Hermite tensor product interpolation polynomials. In particular, we use the existing implementation in deal.II [1] for Lagrange polynomials, and define a basis of Hermite interpolation polynomials [2] $f_i, g_i$ for $i = 0, 1, \ldots, r$ of degree $2r + 1$ on the $1D$ reference interval $[0, 1]$ as follows:

$$\frac{d^j}{dx^j}\left[f_i(x)\right]_{x=0} = 4^i i! \delta_{i,j}, \qquad \frac{d^j}{dx^j}\left[g_i(x)\right]_{x=0} = 0,$$
$$\frac{d^j}{dx^j}\left[f_i(x)\right]_{x=1} = 0, \qquad \frac{d^j}{dx^j}\left[g_i(x)\right]_{x=1} = 4^i i! \delta_{i,j}.$$

In the above the polynomials $f_i$ correspond to derivative degrees of freedom at $x = 0$, and $g_i$ correspond to degrees of freedom at $x = 1$. For Hermite polynomials we enforce continuity of all derivatives up to order $r$ across element boundaries, which leads to fewer degrees of freedom in total for the same polynomial order. These elements are implemented in the deal.II framework to allow a quick comparison with Lagrange elements.

One drawback of Hermite finite elements is the difficulty in enforcing derivative continuity on unstructured meshes. However the intention is to use these elements in an immersed boundary framework where a regular Cartesian grid can be used with a more general domain shape. Another drawback that occurs with the higher order elements is the basis functions do not approximate orthogonality, so the mass matrix is not diagonally dominated and mass-lumping cannot be applied.

For all numerical experiments we consider the wave equation on the spatial domain $\Omega = [0, 3]^2$ and time period $[0, 2)$ :

$$\begin{cases} u_{tt} = u_{xx} + u_{yy}, & (x, y, t) \in \Omega \times (0, 2), \\ u(x, y, 0) = \sin(\pi x)\sin(\pi y), & (x, y) \in \Omega, \\ u_t(x, y, 0) = 0, & (x, y) \in \Omega, \\ u(x, y, t) = 0, & (x, y, t) \in \partial\Omega \times (0, 2), \end{cases}$$

where $\partial\Omega$ denotes the boundary of $\Omega$ and the exact solution is

$$u(x, y, t) = \sin(\pi x)\sin(\pi y)\cos\left(\sqrt{2}\pi t\right).$$

We use the second order leapfrog method to discretise $u_{tt}$ directly, and initialise the system with the exact solution at times $t = 0$ and $t = \delta t$.

## 2 Initial conditions

The first row of the tables gives the dimensions of the spatial grid (16 denotes a $16 \times 16$ grid), and the first column gives the polynomial degree of the finite element basis.

**Lagrange FEM:**

Initial $L^2$-errors over $\Omega$:

| | $N = 16$ | $N = 32$ | $N = 64$ |
|---|---|---|---|
| $p = 1$ | 2.8588e-02 | 6.9298e-03 | 1.7189e-03 |
| $p = 3$ | 5.2059e-05 | 3.1889e-06 | 1.9828e-07 |
| $p = 5$ | 3.7683e-08 | 5.8017e-10 | 9.0409e-12 |

**Hermite FEM:**

Initial $L^2$-errors over $\Omega$:

| | $N = 16$ | $N = 32$ | $N = 64$ |
|---|---|---|---|
| $p = 1$ | 2.8588e-02 | 6.9298e-03 | 1.7189e-03 |
| $p = 3$ | 2.0817e-04 | 1.4095e-05 | 9.0036e-07 |
| $p = 5$ | 1.0568e-07 | 1.5650e-09 | 2.4098e-11 |

Calculating order of accuracy from these results indicates that both methods are $(p + 1)^{\text{th}}$ order accurate in projecting initial conditions.

## 3 Fixed CFL number

We first consider the results when a CFL number of 0.05 is used for all simulation set-ups. This value was found to be numerically stable

for both methods up to polynomial degree $p = 5$. The $L^2$-errors after the final time-step and CPU time in seconds are shown below:

**Lagrange FEM:**

Final $L^2$-errors over $\Omega$:

| | $N = 16$ | $N = 32$ | $N = 64$ |
|---|---|---|---|
| $p = 1$ | 8.7399e-02 | 2.4609e-02 | 6.2750e-03 |
| $p = 3$ | 4.7439e-04 | 1.2220e-04 | 3.0538e-05 |
| $p = 5$ | 4.7049e-04 | 1.2200e-04 | 3.0537e-05 |

Simulation times (CPU seconds):

| | $N = 16$ | $N = 32$ | $N = 64$ |
|---|---|---|---|
| $p = 1$ | 1.7134e+00 | 1.1035e+01 | 7.9023e+01 |
| $p = 3$ | 3.0569e+00 | 1.8632e+01 | 1.3382e+02 |
| $p = 5$ | 6.2698e+00 | 3.8880e+01 | 2.7534e+02 |

**Hermite FEM:**

Final $L^2$-errors over $\Omega$:

| | $N = 16$ | $N = 32$ | $N = 64$ |
|---|---|---|---|
| $p = 1$ | 8.7399e-02 | 2.4609e-02 | 6.2750e-03 |
| $p = 3$ | 5.0804e-04 | 1.2268e-04 | 3.0548e-05 |
| $p = 5$ | 4.7049e-04 | 1.2200e-04 | 3.0537e-05 |

Simulation times (CPU seconds):

| | $N = 16$ | $N = 32$ | $N = 64$ |
|---|---|---|---|
| $p = 1$ | 2.1218e+00 | 1.2389e+01 | 8.7286e+01 |
| $p = 3$ | 3.4353e+00 | 2.0078e+01 | 1.4209e+02 |
| $p = 5$ | 6.4353e+00 | 4.0642e+01 | 2.9798e+02 |

We see that the error at the final time is dominated by the time-stepping error, which is second order. We also note that the errors at the final time and the computation time are very similar for Lagrange and Hermite methods. Interestingly this is despite Hermite having fewer degrees of freedom, indicating more work is needed to find a good preconditioner for the mass matrix.

## 4 Relaxed CFL numbers

As found in [3], it is possible to take significantly larger time-steps for higher order Hermite finite elements than Lagrange while maintaining numerical stability over time. The CFL numbers used for the different methods and polynomial degrees are chosen to be below but not at the stability limit, and are listed below to three significant figures:

| | Lagrange | Hermite |
|---|---|---|
| $p = 1$ | 0.406 | 0.367 |
| $p = 3$ | 0.0979 | 0.184 |
| $p = 5$ | 0.0460 | 0.122 |

All methods remained stable with these CFL numbers. The Lagrange polynomials were tested with a CFL of 0.12 and became unstable for $p = 3, 5$. The difference in CFL for $p = 1$ was purely due to using an approximate formula to generate CFL numbers, and in practice a Hermite method would not be used with $p = 1$.

The final errors for these experiments were again dominated by the time-step errors, which led to worse accuracy for Hermite than Lagrange due to larger time-steps. Methods to improve the accuracy of the Hermite system over time exist but are beyond the scope of this short paper.

**Lagrange FEM:**

Simulation times (CPU seconds):

| | $N = 16$ | $N = 32$ | $N = 64$ |
|---|---|---|---|
| $p = 1$ | 2.4382e-01 | 1.4153e+00 | 1.0287e+01 |
| $p = 3$ | 1.7566e+00 | 9.8094e+00 | 6.8715e+01 |
| $p = 5$ | 6.7261e+00 | 4.1602e+01 | 2.9929e+02 |

**Hermite FEM:**

Simulation times (CPU seconds):

| | $N = 16$ | $N = 32$ | $N = 64$ |
|---|---|---|---|
| $p = 1$ | 5.6322e-01 | 1.7037e+00 | 1.2634e+01 |
| $p = 3$ | 1.4320e+00 | 6.7186e+00 | 4.2364e+01 |
| $p = 5$ | 3.3575e+00 | 1.9046e+01 | 1.3433e+02 |

The larger time-steps that Hermite elements allow produce a significant improvement in computation time compared to Lagrange. This indicates that Hermite elements can offer a significant performance benefit over Lagrange for simulations over long time periods due to the larger maximum stable time-step.

## References

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