Wave propagation in unbounded quasiperiodic media, Part 2: the non-absorbing case

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Abstract

We are interested in the Helmholtz equation in a 1D unbounded quasiperiodic medium (see Part 1 for the absorbing case). We propose a numerical procedure to compute the outgoing solution assuming that a limiting absorption principle holds. The problem is lifted onto a 2D non-elliptic problem with periodic coefficients. However, the method has to be adapted: the Dirichlet-to-Neumann (DtN) coefficients are replaced by Robin-to-Robin (RtR) ones, and with respect to the non-absorbing csase, a condition has to be added to characterize the propagation operator.

Keywords: quasiperiodicity, waveguides

1 Problem setting

We are interested in the Helmholtz equation with frequency $\omega \in \mathbb{R}$:

$$-\big(\mu_\theta\, u'\big)' - \rho_\theta\, \omega^2\, u = f \quad \text{in} \quad \mathbb{R}, \qquad (1)$$

where $f \in L^2(\mathbb{R})$ has a compact support $(-a, a)$, $a > 0$, and where μ_θ and ρ_θ are ***quasiperiodic***, that is, there exists $\theta \in (0, \pi/2)$ and 1–periodic functions μ_p, $\rho_p \in \mathscr{C}^0(\mathbb{R}^2)$ such that

$$\mu_\theta(x) = \mu_p(x\, \vec{e}_\theta) \quad \text{and} \quad \rho_\theta(x) = \rho_p(x\, \vec{e}_\theta). \quad (2)$$

The well-posedness of (1) is unclear. One expects that the physical solution u, if it exists, may not belong to $H^1(\mathbb{R})$ due to a lack of decay at infinity. In this case, one needs a so-called *radiation condition* that imposes the behaviour at infinity. Such a condition can be obtained in practice using the *limiting absorption principle*, which consists in (i) adding some absorption to the problem, and (ii) studying the limit of the solution u as $\mathfrak{Im}\, \omega^2 \to 0$.

2 Mathematical issues

Understanding the limit process described above is closely related to the spectral analysis of the self-adjoint differential operator in $L^2(\mathbb{R}; \rho_\theta\, dx)$:

$$\left|\begin{array}{l} H_\theta u = -\dfrac{1}{\rho_\theta}\,\big(\mu_\theta\, u'\big)', \\ D(H_\theta) = \big\{u \in H^1(\mathbb{R}),\ (\mu_\theta\, u')' \in L^2(\mathbb{R})\big\}. \end{array}\right.$$

When μ_θ and ρ_θ are periodic ie. when $\tan\theta \in \mathbb{Q}$, Floquet theory shows that the spectrum $\sigma(H_\theta)$ is purely continuous with a band structure.

When $\tan\theta$ is irrationnal, $\sigma(H_\theta)$ has an absolutely continuous part as in the periodic case, but may also have a point part, and a even a singular continuous part that may contain a *Cantor set* (that is, a closed set with no isolated points and whose complement is dense, see [2] for related results).

Indeed, there is no problem with the limiting absorption principle when ω^2 is not in $\sigma(H_\theta)$. Of course, it cannot hold when ω^2 is an eigenvalue de H_θ (we exclude this case in the following). In all the other cases, the question is still open.

3 A solution approach

We propose a numerical procedure assuming that the limiting absorption principle holds. One preliminary step is to notice that if $\mathfrak{Im}\, \omega^2 > 0$, then u can be computed by solving problems of the generic form: *Find* $u_\theta \in H^1(\mathbb{R}_+)$ *such that*

$$\left|\begin{array}{c} -\big(\mu_\theta\, u_\theta'\big)' - \rho_\theta\, \omega^2\, u_\theta = 0, \quad \text{in} \quad \mathbb{R}_+^*, \\ u_\theta(0) = 1. \end{array}\right. \qquad (3)$$

Since μ_θ, ρ_θ are traces of periodic functions along $\vec{e}_\theta\, \mathbb{R}$, the idea is to interpret u_θ as the trace along the same line of U_θ, the solution of a 2D periodic problem in $(0, 1) \times \mathbb{R}_+$ with a Dirichlet condition on $(0, 1) \times \{0\}$. The periodicity of the half-guide problem allows us to compute U_θ by solving ***Dirichlet local cell problems*** and by computing the propagation operator $\mathcal{P}$ which is the unique solution of a ***stationary Riccati equation*** with a spectral radius $\rho(\mathcal{P}) < 1$ (more details can be found in Part 1).

The next step then consists in passing to the limit $\mathfrak{Im}\, \omega^2 \to 0$ in the method presented above. Doing so however raises several difficulties. First of all, for $\mathfrak{Im}\, \omega^2 = 0$, considering the Dirichlet half-line problem (3) may introduce artificial edge resonances. More importantly, we have shown that ***the Dirichlet local cell problems are ill-posed for most frequencies*** (i.e. outside an interval). These lead us to introduce the

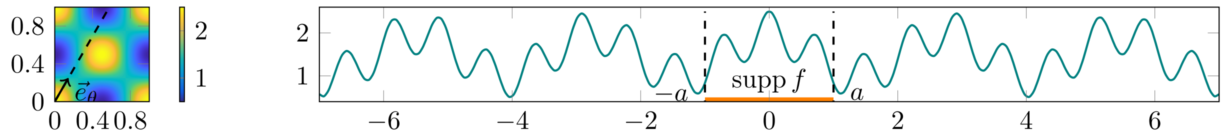

Figure 1: $\mu_p(y_1, y_2) = 1.5 + \cos(2\pi y_1)\,\cos(2\pi y_2)$ (*left*), its quasiperiodic trace along $\vec{e}_\theta$, $\theta = \pi/3$ (*right*)

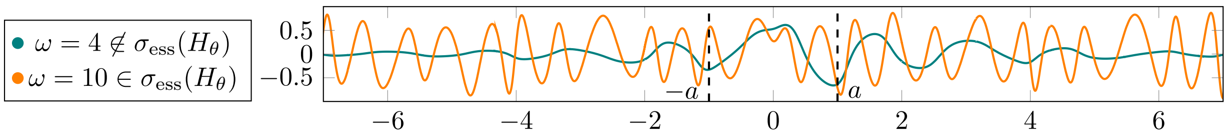

Figure 2: Solution of (1) depending on whether ω^2 belongs or not to $\sigma_{\text{ess}}(H_\theta)$

Robin half-line problem instead of (3):

$$\left|\begin{array}{l} -\left(\mu_\theta\, \widetilde{u}_\theta'\right)' - \rho_\theta\, \omega^2\, \widetilde{u}_\theta = 0, \quad \text{in } \mathbb{R}_+^*, \\ \left[\mu_\theta\, \widetilde{u}_\theta'\right](0) + \mathrm{i}\,\omega\, z\, \widetilde{u}_\theta(0) = 1, \end{array}\right. \tag{4}$$

with $\mathfrak{Re}\, z > 0$, so that the associated operator has no discrete spectrum. We look for solutions $\widetilde{u}_\theta$ as $\widetilde{u}_\theta(x) = \widetilde{U}_\theta(x\,\vec{e}_\theta)$, where $\widetilde{U}_\theta$ satisfies for $(y_1, y_2) \in \Omega := (0,1) \times \mathbb{R}_+^*$ the problem

$$\left|\begin{array}{l} -D_\theta\left(\mu_p\, D_\theta \widetilde{U}_\theta\right) - \rho_p\, \omega^2\, \widetilde{U}_\theta = 0 \quad (\Omega), \\ \sin\theta\, \mu_p\, D_\theta \widetilde{U}_\theta + \mathrm{i}\,\omega\, z\, \widetilde{U}_\theta = \varphi, \quad (y_2 = 0), \\ \widetilde{U}_\theta \text{ is periodic wrt. } y_1 \end{array}\right. \tag{5}$$

with $\varphi \in \mathscr{C}^0(\mathbb{R})$, an arbitrary 1–periodic function that must satisfy $\varphi(0) = 1$ for the sake of consistency with the condition satisfied by $\widetilde{u}_\theta$.

(1) If ω^2 is not in $\sigma_{\text{ess}}(H_\theta)$, the essential spectrum of H_θ, then (5) is well-posed in $H^1_\theta(\Omega) := \{U,\, D_\theta U \in L^2(\Omega)\}$, and the procedure is similar to the absorbing case. More precisely,

$$\widetilde{U}_\theta(\varphi)(y_1, y_2 + \ell) = \widetilde{U}_\theta(\widetilde{\mathcal{P}}^\ell \varphi)(y_1, y_2) \tag{6}$$

where the propagation operator $\widetilde{\mathcal{P}}$ is defined by

$$\widetilde{\mathcal{P}}\varphi = \left[\sin\theta\, \mu_p\, D_\theta \widetilde{U}_\theta(\varphi) + \mathrm{i}\omega z\, \widetilde{U}_\theta(\varphi)\right]\big|_{y_2=1}.$$

In this case, $\widetilde{U}_\theta$ can be computed cell by cell in terms of the solutions E^0, E^1 of ***Robin local cell problems***, i.e. the PDE in (5) completed with periodic conditions in the y_1 direction and Robin conditions (*cf* Figure 3). For any $\omega^2 \in \mathbb{R}$, these local cell problems are well-posed, contrary to the Dirichlet ones. One can show that $\widetilde{\mathcal{P}}$ is the unique solution of a Riccati system with a spectral radius $\rho(\widetilde{\mathcal{P}}) < 1$.

(2) If $\omega^2 \in \sigma_{ess}(H_\theta)$, then (5) is no longer well-posed in $H^1_\theta(\Omega)$. In other terms, the outgoing solution can oscillate without vanishing until

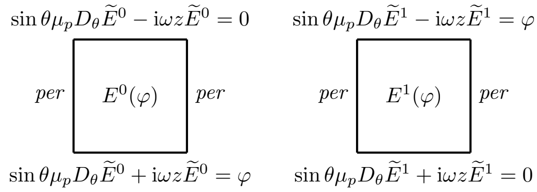

Figure 3: Local cell problems

infinity. In order to construct the outgoing solution, we use the same procedure as in the previous case by computing $\widetilde{E}^0, \widetilde{E}^1$, and by solving the Riccati system. To allow oscillations at infinity for the outgoing solution, one has to look for a solution of the Riccati system of spectral radius equal to 1 (see (6)). However, the Riccati system may admit an infinity of such solutions. To recover uniqueness and characterize the outgoing propagation operator, we adapt the spectral condition proposed in [1]. This condition, obtained by limiting absorption for the classical Helmholtz equation, is linked to the energy flux of the outgoing solution.

Once $\widetilde{\mathcal{P}}$ is obtained, using the solutions of the local cell problems, one can deduce $\widetilde{U}_\theta$ cell by cell and then, provided that ω^2 is not in the discrete spectrum of H_θ, compute coefficients $\lambda_\theta^\pm \in \mathbb{C}$ so that

$$\pm(\mu_\theta u')(\pm a) + \lambda_\theta^\pm u(\pm a) = 0$$

are transparent conditions for (1).

References

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