Speckle statistics in stochastic homogenization regime

Quentin Goepfert[1,*], **Josselin Garnier**[2,2], **Laure Giovangigli**[1], **Pierre Milien**[3,3]

[1]UMA Poems ENSTA, Palaiseau, France
[2]CMAP Ecole Polytechnique, Palaiseau, France
[3]Institut Langevin, Paris, France

*Email: quentin.goepfert@ensta-paris.fr

Abstract

We aim at describing the statistics of the acoustic wavefield backscattered by a randomly heterogeneous penetrable medium in the homogenization regime.

Keywords: stochastic homogenization, wave equation

1 Introduction

In biological ultrasound imaging, the measured quantity is the backscattered wave generated by a large number of unresolved subwavelength scatterers. These scatterers can be modeled as inhomogeneities in density and compressibility. In the Born approximation, the backscattered field is well understood, but the assumption does not hold when the number of scatterers becomes very large, which is the case in many situations. Stochastic homogenization techniques do not rely on single diffusion approximation and can be an accurate model to describe the backscattered field.

2 Framework

Here d denotes the dimension, $d = 1, 2$ or 3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space and let $\mathcal{M} \subset \mathbb{R}^d$ be the medium in which lie small randomly placed inhomogeneities of size $\epsilon > 0$. We denote by $a_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d \times \mathbb{R}^d)$ the density and $n_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^d)$ the compressibility. We suppose that the outer medium $\mathbb{R}^d \setminus \overline{\mathcal{M}}$ is homogeneous of parameters $(I, 1)$. The inhomogeneities $(\epsilon S_i^\omega)_{i\in\mathbb{N}*}$ have constant parameters denoted by $(a_{S_i}^\omega)_{i\in\mathbb{N}*}$ and $(n_{S_i}^\omega)_{i\in\mathbb{N}*}$ and lie in a homogeneous background with properties (a_m, n_m).

For all $x \in \mathbb{R}^d$ and a.e. $\omega \in \Omega$, the density and compressibility are therefore modeled by:

$$\begin{cases} a_\epsilon(x) := a_m(I + \eta_a(x/\epsilon, \omega))\chi_{\mathcal{M}}(x) + (1 - \chi_{\mathcal{M}}(x))I \\ n_\epsilon(x) := n_m(1 + \eta_n(x/\epsilon, \omega))\chi_{\mathcal{M}}(x) + (1 - \chi_{\mathcal{M}}(x))1 \end{cases}$$

with

$$\eta_a(x, \omega) = \sum_{i\in\mathbb{N}*} \left(a_m^{-1} a_{S_i}^\omega - I\right) \chi_{S_i^\omega}(x)$$

$$\eta_n(x, \omega) = \sum_{i\in\mathbb{N}*} \left(\frac{n_{S_i}^\omega}{n_m} - 1\right) \chi_{S_i^\omega}(x)$$

The medium contains around ϵ^{-d} particles, spaced from each other by a distance of order ϵ, so that we are in the homogenization regime. Furthermore we assume that the distribution of the center of the particules $(S_i^\omega)_{i\in\mathbb{N}*}$ of size 1, is a stationary and ergodic process. Therefore η_a and η_n are stationary and ergodic processes. Moreover, a_ϵ and n_ϵ are supposed to be bounded both below and above, by respectively strictly positive matrices or constants independent of the randomness.

We excite the medium by a time-harmonic plane wave u^i. An example is shown on figure 1.

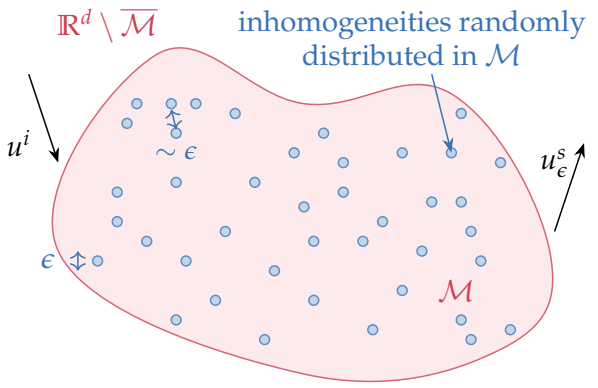

Figure 1: Setup of the scattering problem

The total field $u_\epsilon = u^i + u_\epsilon^s$ is the solution of :

$$\begin{cases} \Delta u_\epsilon^s + k^2 u_\epsilon^s = 0 & \text{in } \mathbb{R}^d \setminus \overline{\mathcal{M}} \\ \nabla \cdot (a_\epsilon \nabla u_\epsilon) + k^2 n_\epsilon u_\epsilon = 0 & \text{in } \mathcal{M} \\ (u_\epsilon^s + u^i) = u_\epsilon & \text{on } \partial\mathcal{M} \\ \nabla(u_\epsilon^s + u^i) \cdot \nu = a_\epsilon \nabla u_\epsilon \cdot \nu & \text{on } \partial\mathcal{M} \end{cases} \tag{1}$$

along with the Sommerfeld radiation condition on the scattered wave field u_ϵ^s.
This problem has already been treated in [3] in the periodic case.

Under the stationary and ergodic assumption, it can be shown [1] that the solution u_ϵ converges in almost surely weakly in $H^1_{loc}(\mathbb{R}^d)$ towards a homogenized field $u_0 = u^i + u_0^s$ solution of :

$$\begin{cases} \Delta u_0^s + k^2 u_0^s = 0 & \text{in } \mathbb{R}^d \setminus \overline{\mathcal{M}} \\ \nabla \cdot (a^\star \nabla u_0) + k^2 n^\star u_0 = 0 & \text{in } \mathcal{M} \\ (u_0^s + u^i) = u_0 & \text{on } \partial\mathcal{M} \\ \nabla (u_0^s + u^i) \cdot \nu = a^\star \nabla u_0 \cdot \nu & \text{on } \partial\mathcal{M} \end{cases} \tag{2}$$

complemented with the radiation condition on u_0^s, for some positive definite and constant tensor $a^\star$ and positive constant $n^\star$.

We will present our estimates of the error in the Hilbert space $\mathcal{L} := L^2(\Omega, L^2_{loc}(\mathbb{R}^d \setminus \overline{\mathcal{M}}))$ in several situations.
Though, in order to get quantitative results, we need to strengthen our hypotheses and assume a quantitative mixing condition [2] such as a Log Sobolev Inequality. Numerous processes satisfy all these conditions such as the type-2 Matérn process which match our model and satisfy a weighted Log-Sobolev Inequality.

3 Case a_ϵ constant

We will first present the case where a_ϵ is constant. Indeed, in this case the equation is simpler and we can find a far field expansion, involving the homogenized Green function G_0, solution for $y \in \mathbb{R}^d$ of:

$$\begin{cases} \Delta G_0(\cdot, y) + k^2 G_0(\cdot, y) = -\delta_y & \text{in } \mathbb{R}^d \setminus \overline{\mathcal{M}} \\ \nabla \cdot (a^\star \nabla G_0(\cdot, y)) + k^2 n^\star G_0(\cdot, y) = -\delta_y & \text{in } \mathcal{M} \end{cases} \tag{3}$$

along with the usual transmission conditions and the Sommerfeld radiation condition.

Theorem 1. *For $x \in \mathbb{R}^d \setminus \overline{\mathcal{M}}$:*

$$u_\epsilon(x) = u_0(x) + k^2 \int_{\mathcal{M}} ((n_\epsilon(y) - n^\star) u_0(y) \times G_0(y, x))\, dy + O_{\mathcal{L}}(\epsilon^d) \tag{4}$$

This result is a generalization of the Born expansion with the homogenized solution instead of the incident wave field. A corollary of Theorem 1 describes the statistics of the wave field $\epsilon^{-d/2} u_\epsilon$ as a zero-mean Gaussian process with a covariance function that depends on the two-point statistics of the random process n_ϵ.

4 1D case

When $d = 1$, explicit formulae can be found and therefore, it is possible to find an asymptotic model in the general case and control all of the error terms.

Theorem 2. *For $x \in \mathbb{R} \setminus \overline{\mathcal{M}}$:*

$$u_\epsilon(x) = u_0(x) + \int_{\mathcal{M}} (\frac{a^\star}{a_\epsilon(y)} - 1) a^\star u_0'(y) \partial_x G_0(x, y) dy + k^2 \int_{\mathcal{M}} (n_\epsilon(y) - n^\star) u_0(y) G_0(x, y) dy + O_{\mathcal{L}}(\epsilon) \tag{5}$$

In dimension 1, we have explicit formulae for $a^\star$ and $n^\star$, that is:

$$a^\star = \mathbb{E}(\frac{1}{a_\epsilon})^{-1} \tag{6}$$

$$n^\star = \mathbb{E}(n_\epsilon) \tag{7}$$

Using these formulae, we can also evaluate the two main terms in (5) which are both of mean zero and of size $\epsilon^{\frac{1}{2}}$ as expected.

5 Conclusion

We have described the behavior of the speckle field generated by a large number of compressibility inhomogeneties which results in an analogue to the Born approximation. The case when the inhomogeneities also present a density contrast shall be treated in forthcoming work.

References

[1] A. Gloria, S. Neukamm and F. Otto, Quantitative estimates in stochastic homogenization for correlated coefficient fields (2019).

[2] M. Duerinckx and A. Gloria, Multiscale functional inequalities in probability: Constructive approach *arXiv preprint arXiv:1711.03152*,(2017).

[3] C. Fioralba, B. Guzina and S. Moskow, On the Homogenization of a Scalar Scattering Problem for Highly Oscillating Anisotropic Media, *SIAM J. Math. Anal.* **48** (2016), pp. 2532-2560.