An Efficient Iterative High Order Numerical Method for Multiple Scattering

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Abstract

We have developed a highly accurate, efficient, and high order numerical iterative method for multiple scattering by coupling finite differences (FD) and a isogeometric finite element method (IGA) with local high order ABC, based on Karp's farfield expansions.

Keywords: Multiple Scattering, High Order Method, Absorbing Boundary Conditions

1 Formulation of the Iterative Multiple Scattering Problem Combined with Karp's Farfield Expansion ABC

Our problem consists of finding the scattered wave, u, obtained from the scattering of a time-harmonic incident plane wave, u_{inc}, from M multiple obstacles. The boundary of each obstacle is denoted as Γ_m and the unbounded region in the exterior of Γ_m by Ω_m, for $m = 1, \ldots M$. Therefore, the common scattered region is $\Omega = \bigcap_{m=1}^{M} \Omega_m$ and the boundary of the full problem is $\Gamma = \bigcup_{m=1}^{M} \Gamma_m$. Hence, our problem can be formulated as

$$\triangle u + k^2 u = 0, \qquad \text{in } \Omega, \tag{1}$$

$$\mathcal{B}u = Z\frac{\partial u}{\partial \nu} + (1-Z)u = \tag{2}$$

$$- Z\frac{\partial u_{inc}}{\partial \nu} + (1-Z)u_{inc}, \quad \text{in } \Gamma,$$

$$\lim_{r\to\infty} r^{1/2}(\partial_r u - iku) = 0, \tag{3}$$

where Z represents the acoustic hardness $Z \in [0,1]$. In this work, we adopt a formulation, proposed by Guizane et al. [1], which consists of considering M single scattering problems in each Ω_m coupled at their scatterer boundaries. Each scattered wave u_m is obtained from the scattering of the incident wave, u_{inc}, and the waves $u_{\bar{m}}$ emanating from the other scatterers ($\bar{m} \neq m$). Balabane [4] proved that the solution u of the multiple scattering problem can be uniquely decomposed as, $u = \sum_{m=1}^{M} u_m$, in Ω. As a consequence, we approximate u by numerically solving a family of single scattering problems for u_m coupled at their scatterer boundaries, Γ_m. Since our purpose is to apply finite differences and finite element methods, we introduce a circular artificial boundary $\mathcal{C}_m$ of radius R^m enclosing the m-th obstacle, which reduces each infinite domain Ω_m to a bounded one, Ω_m^-. Next, we define an appropriate ABC on $\mathcal{C}_m$. For this purpose, we employ a high order local ABC derived by Villamizar, Acosta and Dastrup [2], and adopted by Khajah et al., [3] to derive an overall high order technique, based on IGA.

Governing equations and scatterer boundary conditions for obstacle m:

$$\Delta u_m + k^2 u_m = 0, \quad \text{in } \Omega_m^-, \tag{4}$$

$$\mathcal{B}u_m = -\mathcal{B}\left(u_{inc} + \sum_{\substack{\bar{m}=1\\ \bar{m}\neq m}}^{M} u_{\bar{m}}\right) \quad \text{on } \Gamma_m. \tag{5}$$

High order local farfield expansion ABC on the circular artificial boundary, $\mathcal{C}_m$ of radius R_m:

$$u_m(R_m,\theta_m) = \mathcal{K}_{m,L}(R_m,\theta_m)), \quad \text{in } \mathcal{C}_m, \tag{6}$$

$$\frac{\partial u_m}{\partial r_m}(R_m,\theta_m) = \frac{\partial \mathcal{K}_{m,L}}{\partial r_m}(R_m,\theta_m) \quad \text{in } \mathcal{C}_m, \tag{7}$$

$$H_0(kR_m)\Big[(L-1)^2 F_{m,L-1} + d^2_{\theta_m} F_{m,L-1}\Big] +$$

$$H_1(kR_m)\Big[L^2 G_{m,L-1} + d^2_{\theta_m} G_{m,L-1}\Big] = 0, \tag{8}$$

$$2l\, G_{m,l} = (l-1)^2 F_{m,l-1} + d^2_{\theta^m} F_{m,l-1} \tag{9}$$

$$2l\, F_{m,l} = -l^2 G_{m,l-1}(\theta) - d^2_{\theta} G_{m,l-1}, \tag{10}$$

on $\mathcal{C}_m$, for $m = 1, \ldots M$ and $l = 1 \ldots L-1$,

where θ^m is the angle along $\mathcal{C}_m$. The symbol $\mathcal{K}_{m,L}$, designates a truncated version of the Karp farfield expansion,
$\mathcal{K}_{m,L}(r_m,\theta_m) = H_0(kr^m)\sum_{l=0}^{L-1} \frac{F_{m,l}(\theta_m)}{(kr_m)^l} + H_1(kr_m)\sum_{l=0}^{L-1} \frac{G_{m,l}(\theta_m)}{(kr_m)^l}$, in Ω_m^+,
which is an exact representation of the solution u_m in the unbounded exterior region $\Omega_m^+ = \mathbb{R}^2 \setminus \overline{\Omega_m^-}$. The angular functions $F_{m,l}(\theta_m)$ and

$G_{m,l}(\theta_m)$ are additional unknowns. They depend on the geometry of the scatterers and the physical properties of the bounded regions Ω_m^-. Once that approximations for $F_{m,l}$ and $G_{m,l}$ are obtained, we can compute the values of u_m anywhere in the exterior region Ω_m^+, in particular, at the other scatterers boundaries.

The computation of the solutions u_m of the family of multiple scattering problems (4)-(10) can be greatly simplified by adopting an iterative Jacobi- or Gauss-Seidel-type formulation, similar to the one found in [1]. Assuming that the outgoing waves from other scatterers are known from a previous iteration, it is possible to reduce the multiple scattering problem (4)-(10) to a family of single scattering problems. They can be solved individually in their respective local coordinate systems to obtain u_m and then adding these u_m, we obtain the total scattered field u from all scatterers.

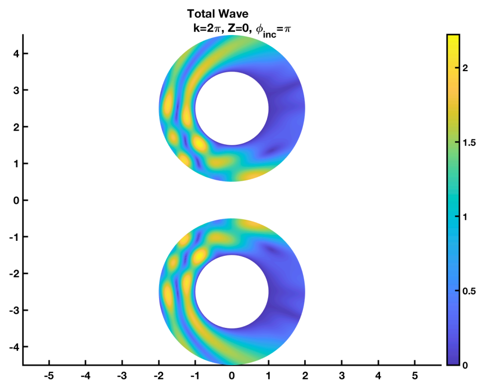

Figure 1: Total field numerical solution

2 Numerical Results

We obtain numerical approximations to the original multiple scattering problem by applying two different numerical methods to the iterative problems: 1) a second order finite difference, and 2) a high order isogeometric finite element method. A key aspect of the proposed numerical techniques is that each of the ultimate matrices A_m, which are obtained from the numerical method (either FD or IGA) when solving the BVP for u_m, do not change during all iteration-steps. Hence, only one matrix inversion is needed during the iteration process. This represents a significant saving of computational time. In Table 1, we show the second order of convergence expected for our second order FD method. As seen, it requires very few iterations. In Table 2, we show a much higher accuracy for the same grid sizes. Also, the higher order of convergence of the IGA method combined with the farfield ABC is clearly shown. In both cases, we are only using 10 terms in the Karp's farfield expansion. We have performed a great number of numerical experiments including complex configuration of several obstacles of arbitrary shape that show the high accuracy, high order of conevergence, and efficiency of our numerical methods. We will discuss them during our presentation at Waves 2022.

Table 1: Finite Differences Converg. Analysis

PPW	h	Iters	Error	Order
15	6.67-02	8	8.93-03	—
20	5.00-02	6	4.71-03	2.23
25	4.00-02	5	2.91-03	2.15
30	3.33-02	5	1.98-03	2.12

Table 2: IGA (p=3) Convergence Analysis

PPW	h	Iters	Error	Order
15	6.67-02	16	9.27-07	—
20	5.00-02	16	2.61-07	4.33
25	4.00-02	16	1.05-07	4.03
30	3.33-02	16	5.39-08	3.60

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