The Time Domain Linear Sampling Method for Maxwell's equations

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Abstract

We investigate the use of the Time Domain Linear Sampling Method (TD-LSM) for determining the shape of a scatterer using electromagnetic waves. Assuming that the scatterer is impenetrable with an impedance boundary condition, we use the Fourier-Laplace transform approach to justify the TD-LSM. We provide numerical results for the impedance boundary condition, for a perfectly conducting obstacle, and for a penetrable body. The latter case cannot be analyzed by the Fourier-Laplace approach, yet still shows good reconstruction quality.

Keywords: Inverse problem, electromagnetism, impedance, linear sampling, time domain.

1 Introduction

The Time Domain Linear Sampling Method (TD-LSM) is an extension of the original frequency domain Linear Sampling Method to the time domain. Previous tests with the acoustic wave equation have shown that the method can be applied with the sound soft or impedance boundary conditions, as well as for penetrable objects. It can also be used with less source and measurement points than for a single frequency linear sampling method [2, 3].

We shall describe an extension of the TD-LSM method to the time dependent Maxwell system governing scattering from an impenetrable scatterer with an impedance boundary condition [4]. We prove that the usual theorems underlying the TD-LSM hold in this case, and provide numerical examples of the method. It is interesting to note that current proofs based on the Fourier-Laplace transform cannot be used for the penetrable case [1], even though we shall give numerical results that show that the method does perform as expected in this case.

2 The forward problem

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ with unit outward normal $\boldsymbol{\nu}$ on $\Gamma = \partial\Omega$, and setting $\Omega^c = \mathbb{R} \setminus \overline{\Omega}$, the electromagnetic field $(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}})$ satisfies

$$\begin{aligned} c_0^{-1}\, \boldsymbol{\mathcal{E}}_t^s - \mathbf{curl}\, \boldsymbol{\mathcal{H}} &= \mathbf{0} \qquad \text{in } \boldsymbol{\Omega^c} \text{ for } \mathbf{t} > \mathbf{0}\,, \\ c_0^{-1}\, \boldsymbol{\mathcal{H}}_t + \mathbf{curl}\, \boldsymbol{\mathcal{E}} &= \mathbf{0} \qquad \text{in } \boldsymbol{\Omega^c} \text{ for } \mathbf{t} > \mathbf{0}\,, \end{aligned}$$

where c_0 is the speed of light. The field is subject to the initial conditions

$$\boldsymbol{\mathcal{E}} \;=\; \boldsymbol{\mathcal{E}}_t = 0 \text{ for } t = 0 \text{ in } \Omega^c\,,$$

and the impedance boundary condition

$$\boldsymbol{\mathcal{H}} \times \boldsymbol{\nu} + \Lambda \boldsymbol{\mathcal{E}}_T \;=\; -(\boldsymbol{\mathcal{H}}^i \times \boldsymbol{\nu} + \Lambda \boldsymbol{\mathcal{E}}_T^i)\,,$$

on Γ for $t > 0$. Here the subscript T denotes the tangential trace here taken on Γ, and $\Lambda > 0$ is the impedance constant. The incident field $\boldsymbol{\mathcal{E}}^i$ is given by a regularized magnetic dipole situated at $\mathbf{y} \in \mathbb{R}^3$ with polarization $\boldsymbol{p} \in \mathbb{R}^3$:

$$\mathcal{E}^i(\mathbf{x}, t; \mathbf{y}, \boldsymbol{p}) = -\boldsymbol{p} \times \nabla_{\mathbf{x}} \Phi_\chi(\mathbf{x} - \mathbf{y}, t)\,.$$

Here

$$\Phi_\chi(\mathbf{x}, t) = \frac{\chi(t - c_0^{-1}|\mathbf{x}|)}{4\pi|\mathbf{x}|}$$

and $\chi \in C_0^\infty((0, \infty))$ is a smooth function of time. We prove that the above problem has a unique solution, $\boldsymbol{\mathcal{E}} := \boldsymbol{\mathcal{E}}(\mathbf{x}, t; \mathbf{y}, \boldsymbol{p})$ for $t > 0$ and note that $\boldsymbol{\mathcal{E}}$ is linear in $\boldsymbol{p}$.

3 The inverse problem

We assume the existence of a surface Σ containing Ω in its interior. Then the inverse problem we wish to solve is to reconstruct the boundary of the scatterer Γ from a knowledge of $\mathcal{E}(\mathbf{x}, t; \mathbf{y}, \boldsymbol{p})$ for $\mathbf{x}, \mathbf{y} \in \Sigma$, $t > 0$ and $\boldsymbol{p} \in \mathbb{R}^3$. To do this we define the *near field* operator applied to a suitable tangential space-time vector function $\mathbf{f}$ by

$$(\mathcal{N}_\chi \mathbf{f})(\mathbf{x}, t) \;=\; \int_{\mathbb{R}} \int_\Sigma \mathcal{E}_T(\mathbf{x}, t-\tau; \mathbf{y}, \mathbf{f}(\mathbf{y}, \tau))\, dA_{\mathbf{y}}\, d\tau$$

for $(\mathbf{x}, t) \in \Sigma \times \mathbb{R}$, where the subscript T refers to the tangential trace here taken on Σ.

Using the near field operator, we seek an approximate solution $\mathbf{g} = \mathbf{g}(\cdot, \cdot; \mathbf{z}, \boldsymbol{p}_\mathbf{z}, \tau_\mathbf{z})$ in an appropriate weighted space-time Sobolev space of the *near field equation*

$$\big(\mathcal{N}_\chi \mathbf{g}(\cdot, \cdot; \mathbf{z}, \boldsymbol{p}_\mathbf{z}, \tau_\mathbf{z})\big)(\mathbf{x}, t) = \mathcal{E}_T^i(\mathbf{x}, t - \tau; \mathbf{z}, \mathbf{q})$$

for $(\mathbf{x}, t) \in \Sigma \times \mathbb{R}$. Here τ is a fixed time translation parameter. This is an ill-posed problem, but we can prove that an approximate solution exists. We also show a version of the usual theorem of blowup of a suitable norm of $\mathbf{g}$ when $\mathbf{z} \in \Omega^c$ [4].

4 Numerical results

We choose a Ricker wavelet for χ:

$$\chi(t) = -\left(1 + 2a\,(t - t_0)^2\right) \exp\left(a\,(t - t_0)^2\right),$$

where $a = -(\pi f_0)^2$, is used as the source modulation function for the incident field. Here f_0 is the peak frequency of the source, and $t_0 = 1.2/f_0$ is the time delay. For the example below, we choose $f_0 = 1$, and use 96 measurements and 54 sources on this surface. The surface Σ is the boundary of the cube $[-4, 4]^3$ and we use 96 measurement points and 54 source points on this surface. Using a nodal discontinuous Galerkin scheme, we compute the solution of the forward problem and hence the near field operator.

By solving a discrete analogue of the near field equation using Tikhonov regularization for $\mathbf{z}$ in a $41 \times 41 \times 41$ grid in the search domain $[-1.5, 1.5]^3$, we can graph isosurfaces of a suitable norm of $\mathbf{g}$ and hence visualize the scatterer. Preliminary results for two cubes are shown in Fig. 1. We show results for an impedance boundary condition with $\Lambda = \sqrt{2}$, a perfectly electrically conducting boundary condition and a penetrable object (with $\epsilon_r = 2$ in Ω).

5 Conclusion

Our preliminary results show that the TD-LSM can be used to identify the shape of scatterers with a variety of boundary conditions. Future work will include a detailed study of the Maxwell TD-LSM with differerent source and measurement geometries, and more complex scatterers.

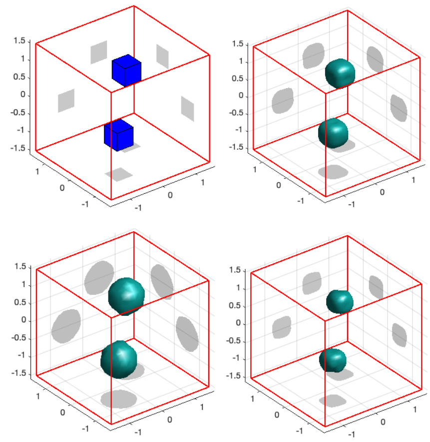

Figure 1: Top row: left panel shows the exact scatterer, right panel shows results for impedance boundary conditions. Bottom row: left panel shows results for PEC boundary conditions, and right panel shows results for a penetrable scatterer. The red cube shows the search region containing the sampling points $\mathbf{z}$.

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