The inflection point problem - from modal to scattering behaviour

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Abstract

The “inflection point problem”, describing the local asymptotic behaviour of high frequency wave fields near an inflection point in the boundary of an obstacle, is a key unsolved canonical problem in wave scattering. In this talk we provide a historical perspective on the problem and describe recent work relating to it.

Keywords: Canonical problem, high frequency asymptotics, parabolic wave equation

In the high frequency/short wavelength regime, it is known that smooth concave boundaries can support so-called “whispering gallery” waves, which propagate along the boundary and are localised close to it. When such a whispering gallery wave encounters an inflection point in the boundary, one expects it to break up, generating exponentially decaying “creeping waves” on the subsequent convex portion of the boundary, along with a “searchlight beam” propagating away from the boundary (see Figure 1 for an illustration). However, determining the amplitude of the creeping waves and searchlight beam from the amplitude of the incoming whispering gallery wave remains an important open problem in scattering theory. One can argue that this should be considered a fundamental problem in PDE theory, since it connects asymptotically “modal” behaviour and “scattering” behaviour.

To describe the problem mathematically, consider the Helmholtz equation $\Delta u + k^2 u = 0$ near a Dirichlet boundary Γ with an inflection point O. Let (s, n) be local curvilinear coordinates representing arclength along and normal distance from Γ, with $s = n = 0$ at O, and suppose the curvature of Γ satisfies $\kappa(s) = -s+O(s^2)$ as $s \to 0$. To investigate the high frequency behaviour near O we introduce stretched coordinates $x = k^{3/5}n$ and $t = k^{1/5}s$ and seek a solution of the form

$$u \sim C(k)\mathrm{e}^{\mathrm{i}ks}\psi(x,t), \quad k \to \infty,$$

for an appropriate k-dependent normalisation constant $C(k)$ (not discussed further here). Inserting this ansatz into the Helmholtz equation reveals that the function $\psi(x, t)$ satisfies

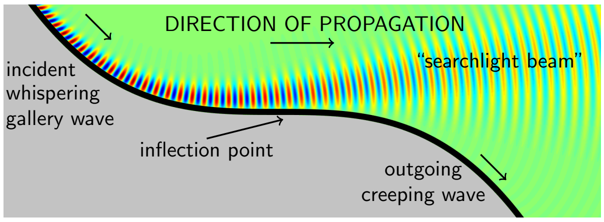

Figure 1: The inflection point solution, computed using the numerical method of [2].

$$\mathrm{i}\frac{\partial\psi}{\partial t} + \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + xt\psi = 0, \quad x > 0, t \in \mathbb{R}, \quad (1)$$

$$\psi(0,t) = 0, \quad t \in \mathbb{R}, \quad (2)$$

and to match with an incoming whispering gallery wave on the concave portion of Γ we require

$$\|\psi(\cdot,t) - \psi_0^-(\cdot,t)\|_{L^2(0,+\infty)} \to 0, \; t \to -\infty, \; (3)$$

where, with $-\nu_j$ denoting a zero of Ai,

$$\psi_0^-(x,t) \;=\; (-2t)^{\frac{1}{6}}\mathrm{e}^{i\nu_j\frac{3}{20}(-2t)^{\frac{5}{3}}}\,\mathrm{Ai}\left(x(-2t)^{\frac{1}{3}} - \nu_j\right).$$

The problem can be equivalently formulated by introducing a new variable $\zeta = x - \frac{1}{6}t^3$, so that (t, ζ) represent approximate local Cartesian coordinates near O, and writing

$$\psi(x,t) = \mathrm{e}^{\frac{\mathrm{i}}{2}\zeta t^2 + \frac{7\mathrm{i}}{120}t^5}\varphi(\zeta,t), \quad (4)$$

to find that $\varphi(\zeta, t)$ satisfies the parabolic wave equation (free Schrödinger equation)

$$\mathrm{i}\frac{\partial\varphi}{\partial t} + \frac{1}{2}\frac{\partial^2\varphi}{\partial\zeta^2} = 0, \quad \zeta > -\frac{1}{6}t^3, \; t \in \mathbb{R}, \quad (5)$$

$$\varphi\big(-\frac{1}{6}t^3, t\big) = 0, \quad t \in \mathbb{R}, \quad (6)$$

with a homogeneous boundary condition on a cubic parabola $\zeta = -\frac{1}{6}t^3$, and an appropriate modification of (3) at $t = -\infty$.

The problem (1)-(3) was first formulated by Popov in [1], and has since been studied by a number of authors. Popov and Pshenchik derived and analysed a finite difference method for the numerical solution of the problem in [2], results generated from which are shown in Figure 1. In [3], Smyshlyaev and Babic proved well-posedness of the problem in an appropriate functional setting, along with certain regularity and decay properties of the solution.

In recent work [4], Smyshlyaev and Kamotski rigorously proved the existence of a unique "searchlight amplitude" $G_0(\eta) \in H^1(\mathbb{R})$ with $\eta G_0(\eta) \in L^2(\mathbb{R})$, such that

$$\left\|\psi(\cdot,t) - \psi_0^+(\cdot,t)\right\|_{L^2(0,+\infty)}(t) \to 0, \quad \text{as } t \to +\infty, \tag{7}$$

where, with $\eta := \frac{\zeta}{t} = \frac{x}{t} - \frac{1}{6}t^2$ and $t > 0$,

$$\psi_0^+(x,t) := t^{-\frac{1}{2}} \mathrm{e}^{\frac{7\mathrm{i}}{120}t^5 + \frac{i}{2}\eta t^3 + \frac{i}{2}\eta^2 t} G_0(\eta). \tag{8}$$

However, providing a closed-form expression for $G_0(\eta)$ remains an open problem.

There have been numerous attempts to derive analytical solutions to (1)-(3), but so far all have proven unsuccessful. In particular, the Fourier transform approach used to derive the solution to the simpler Fock-Leontovich tangent ray diffraction problem (see e.g. [5]) fails here, because the problem (1)-(3) is non-separable.

In [6,7] contour integral representations were considered for the solution of (5) in the form

$$\varphi(\zeta,t) = \int_\gamma F(z) \mathrm{e}^{\mathrm{i}\left(-\zeta z^2 - \frac{t}{2}z^4 + \frac{4\sqrt{2}}{15}z^5\right)} \,\mathrm{d}z, \tag{9}$$

where Γ is a suitable contour in the complex z-plane, starting and ending at infinity, and $F(z)$ is analytic in a neighbourhood of γ. Such integrals are related to the classical catastrophe integrals described e.g. in [8], and to a recently-discovered representation for the solution of the Fock-Leontovich problem (see [5]).

In [7] it was shown using a steepest descent (saddle point) analysis that the function $F(z)$ and contour γ can be chosen in such a way as to asymptotically recover *either* whispering gallery waves as $t \to -\infty$, or creeping waves as $t \to +\infty$. But as yet no choice of $F(z)$ and γ has been found to produce both simultaneously.

Most recently [9], Naqvi and Smyshlyaev have investigated boundary integral equation reformulations of (1)-(3) that appear to shed new light on the problem both analytically and numerically. One exciting avenue of possible future research could be the development of accurate numerical methods for the solution of the boundary integral equations derived in [9], perhaps even using a hybrid numerical-asymptotic approach that incorporates the anticipated asymptotic structure of the solution.

References

[1] Popov MM, The problem of whispering gallery waves in a neighbourhood of a simple zero of the effective curvature of the boundary, *J Sov Math*, 11 (1979), 791–797.

[2] Popov MM and Pshenchik, I, Numerical solution of the problem on whispering gallery waves in a neighborhood of a simple zero of the effective curvature of the boundary, *J Sov Math*, 11 (1979), 797–804.

[3] Babich VM, Smyshlyaev VP, The scattering problem for the Schrödinger equation with a potential linear in time and in space. II: Correctness, smoothness, behavior of the solution at infinity, *J Sov Math*, 38 (1987), 1562–1576.

[4] Smyshlyaev VP, Kamotski IV, Searchlight asymptotics for high-frequency scattering by boundary inflection, *St Petersburg Math J*, to appear. [*Algebra i Analiz* 33 (2021), 275–297; https://arxiv.org/abs/2103.04734]

[5] Hewett, DP, Tangent ray diffraction and the Pekeris caret function, *Wave Motion*, 57 (2015), 257–267.

[6] Kazakov AY, Special function related to the concave-convex boundary problem of the diffraction theory, *J Phys A: Math Gen* 36 (2003), 4127–4141.

[7] Hewett DP, Ockendon JR, Smyshlyaev VP, Contour integral solutions of the parabolic wave equation, *Wave Motion*, 84 (2019), 90–109.

[8] Berry, MV and Howls, CJ, Chapter 36 Integrals with Coalescing Saddles, *NIST Digital Library of Mathematical Functions*, http://dlmf.nist.gov/

[9] Naqvi S, High frequency whispering gallery waves near boundary inflection, *MSci Thesis*, University College London, 2021.