Locally Implicit Preconditioning for Maxwell equations on a locally refined spatial grid

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Abstract

In this talk, we discuss the construction and analysis of higher-order time integration schemes for the full discretization of linear Maxwell equations on locally refined spatial grids. The schemes are based on a higher-order implicit method, e.g., an algebraically stable Runge–Kutta method. Our main contribution is to propose a preconditioned Krylov subspace method for solving the linear systems arising in each time step, which is designed in such a way that its convergence only depends on the coarse mesh but not on the fine mesh. This is shown by approximation theory in the complex plane.

The advantage of this approach is that it is applicable to any implicit scheme and also works for exponential integrators. It is even applicable to nonlinear problems, where such linear systems arise within the Newton iterations.

Keywords: locally refined spatial grid, Maxwell equations, higher-order time integration, Krylov subspace methods, preconditioning.

1 Introduction

Let $\Omega \subset \mathbb{R}^d, d = 1, 2, 3$, be an open, bounded Lipschitz domain and $T > 0$ be the simulation time. The linear Maxwell equations in a medium with permeability $\mu : \Omega \to \mathbb{R}$, permittivity $\epsilon : \Omega \to \mathbb{R}$, and a perfect conducting boundary are given by

$$\begin{aligned} \mu\partial_t \mathbf{H} &= -\operatorname{curl} \mathbf{E}, && (0,T)\times\Omega, \\ \epsilon\partial_t \mathbf{E} &= \operatorname{curl} \mathbf{H} - \mathbf{J}, && (0,T)\times\Omega, \\ \mathbf{H}(0) &= \mathbf{H}^0, \quad \mathbf{E}(0) = \mathbf{E}^0, && \Omega, \\ \mathbf{n}\times\mathbf{E} &= 0, && (0,T)\times\partial\Omega. \end{aligned} \tag{1}$$

Here, $\mathbf{H}, \mathbf{E}, \mathbf{J} : (0,T)\times\Omega \to \mathbb{R}^d$ denote the unknown magnetic and electric field, and the given current density, respectively. The vector $\mathbf{n}$ denotes the unit outward normal vector of the domain Ω. Discretization of (1) in space using a dG method with central flux [5, Section 2] leads to

$$\partial_t \mathbf{u}_h(t) = \mathcal{C}\mathbf{u}_h(t) + \mathbf{j}_h(t),\ \mathbf{u}_h^0 = \mathbf{u}_h(0), \tag{2}$$

where

$$\mathbf{u}_h = \begin{bmatrix} \mathbf{H}_h \\ \mathbf{E}_h \end{bmatrix}, \mathcal{C} = \begin{pmatrix} 0 & -\mathcal{C}_E \\ \mathcal{C}_H & 0 \end{pmatrix}, \mathbf{j}_h = \begin{bmatrix} 0 \\ -\mathbf{J}_h \end{bmatrix}.$$

Here, $\mathcal{C}_H$ and $\mathcal{C}_E$ are spatially discretized curl-operators. The boundary condition for the electric field is weakly enforced in the definition of $\mathcal{C}_E$.

We split the locally refined mesh $\mathcal{T}_h$ into a coarse mesh $\mathcal{T}_{h,c}$ and a fine mesh $\mathcal{T}_{h,f}$ with minimum mesh sizes h_c and h_f, respectively. The methods are attractive if $h_f \ll h_c$ and $\operatorname{card}(\mathcal{T}_{h,f}) \ll \operatorname{card}(\mathcal{T}_{h,c})$. Based on this decomposition of the mesh, the split discrete curl operators $\mathcal{C}_H^e, \mathcal{C}_H^i, \mathcal{C}_E^e, \mathcal{C}_E^i$ defined in [5, Definition 2.7] satisfy

$$\mathcal{C}_H = \mathcal{C}_H^i + \mathcal{C}_H^e, \qquad \mathcal{C}_E = \mathcal{C}_E^i + \mathcal{C}_E^e \tag{3}$$

and

$$\mathcal{C}_H^e\mathcal{C}_E^e = \mathcal{C}_H^e\mathcal{C}_E, \qquad \mathcal{C}_H^i\mathcal{C}_E^i = \mathcal{C}_H^i\mathcal{C}_E. \tag{4}$$

In fact, it was shown in [5], that not only the fine elements have to be treated implicitly but also their coarse neighbors. Then, the split operators $\mathcal{C}_H^e$ and $\mathcal{C}_E^e$ can be bounded independently of fine mesh sizes h_f, i.e.,

$$\|\mathcal{C}_E^e\| \le c h_c^{-1}, \quad \text{and} \quad \|\mathcal{C}_H^e\| \le c h_c^{-1}, \tag{5}$$

with a constant c that is independent of h_f and h_c.

2 Higher-order time integration

The efficient implementation of an s-stage Gauss Runge-Kutta method [3, Section II.1] for the time integration of (2) with step size $\tau > 0$ requires solving linear systems of equations of the form

$$\mathcal{A}\mathbf{x} = \mathbf{b}, \quad \text{where} \quad \mathcal{A} := \mathcal{I} + \tau^2\alpha\mathcal{C}_H\mathcal{C}_E \tag{6}$$

in each time step. The real or complex parameter $\alpha := \alpha_R + i\alpha_I$ only depends on the coefficients of the Runge-Kutta method but neither on the problem nor on the mesh. Since $\mathcal{C}_H\mathcal{C}_E$ is real and symmetric, $\mathcal{A}$ is complex symmetric.

3 Krylov subspace methods

To exploit the structure of $\mathcal{A}$, we suggest to use the quasi-minimal residual (QMR) algorithm for complex symmetric matrices [2, Section 3], which is based on the complex symmetric Lanczos process. For an initial guess $\mathbf{x}_0$ and an initial residue $\mathbf{r}_0$, QMR yields an approximation $\mathbf{x}_m \in \mathbf{x}_0 + \mathcal{K}_m(\mathcal{A}, \mathbf{r}_0)$, where $\mathcal{K}_m(\mathcal{A}, \mathbf{r}_0)$ is Krylov subspace generated by $\mathcal{A}$ and $\mathbf{r}_0$.

Analogously to [4, Theorem 2], one can prove that the error of the QMR iterates satisfies

$$\left\|\mathcal{A}^{-1}\mathbf{b} - \mathbf{x}_m\right\| \leq C \min_{\substack{p_m \in \mathcal{P}_m \\ p_m(0)=1}} \|p_m(\mathcal{A})\| \, \|\mathbf{r}_0\| \,, \quad (7)$$

with a constant C independent of $\|\mathcal{A}\|$. Here, $\mathcal{P}_m$ denotes the set of all polynomials of degree at most m. If the field of values $\mathcal{F}(\mathcal{A})$ is contained in a convex, bounded set $\mathcal{S}$, then, using Faber polynomials and complex approximation theory, cf. [1], we have

$$\|p_m(\mathcal{A})\| \leq (1+\sqrt{2}) \max_{z \in \mathcal{S}} |p_m(z)| \,. \quad (8)$$

Note that the largest elements in $\mathcal{F}(\mathcal{A})$ are of the order h_f^{-1}.

4 Preconditioning

Obviously, a smaller set $\mathcal{S}$ in (8) leads to faster convergence. To speed up the convergence, we aim to construct a preconditioner such that the field of values of the preconditioned matrix can be bounded independently of h_f. Motivated by locally implicit methods for Maxwell equations in [5], we suggest to approximate $\mathcal{A}$ by its dominant part,

$$\mathcal{A} \approx \mathcal{B} := \mathcal{I} + \tau^2 \rho \mathcal{C}_H^i \mathcal{C}_E^i, \quad \rho > 0, \quad (9)$$

i.e., we replace the discrete curl operators $\mathcal{C}_H, \mathcal{C}_E$ in (6) defined on the full mesh by the split operators acting on the implicitly treated mesh elements and α by a real parameter $\rho > 0$. Hence $\mathcal{B}$ is a symmetric, positive definite matrix, which allows us to define the equivalent preconditioned linear system by

$$\widetilde{\mathcal{A}}\widetilde{\mathbf{x}} = \widetilde{\mathbf{b}}, \quad \widetilde{\mathcal{A}} := \mathcal{B}^{-1/2}\mathcal{A}\mathcal{B}^{-1/2}, \quad (10)$$

where $\widetilde{\mathbf{x}} := \mathcal{B}^{1/2}\mathbf{x}$ and $\widetilde{\mathbf{b}} := \mathcal{B}^{-1/2}\mathbf{b}$. Since $\mathcal{A}$ is complex symmetric and $\mathcal{B}$ is real symmetric, the preconditioned matrix $\widetilde{\mathcal{A}}$ is again complex symmetric. We now apply the complex symmetric QMR method to the preconditioned linear system (10) and refer to this method as preconditioned QMR method (pQMR). For $\mathcal{F}(\widetilde{\mathcal{A}})$ we have proven the following theorem.

Theorem 1 *The field of values of $\widetilde{\mathcal{A}}$ defined in (10) satisfies $\mathcal{F}(\widetilde{\mathcal{A}}) \subset \widetilde{\mathcal{S}}$, where $\widetilde{\mathcal{S}}$ is bounded independently of the fine mesh.*

By (7) and (8), this theorem shows that the convergence of the preconditioned QMR method is indeed independent of the fine mesh.

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