Spectral analysis of generalized normal modes

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Abstract

We study a spectral problem defined by an unforced Helmholtz equation when using the permittivity as the eigenvalue. The eigensolutions of such problems, so-called generalised normal modes (GNM), have been recently applied by different authors for modal expanding electromagnetic scattering fields by resonant nanocavities in nanophotonic systems. In this work we make progress in the theoretical underpinnings of GNM by proving their completeness in a pertinent energy functional space. We further prove that they form a Riesz basis in some particular configurations.

Keywords: Electromagnetic resonances, modal expansion, spectral theory, non-selfadjoint operators.

1 Introduction

An efficient approach to the analysis of the electromagnetic field scattered by a nanoresonator subject to radiation losses (i.e. in an open system), is to proceed by a modal expansion approach wherein the resonant response can be described as an infinite sum using as basis functions the eigenfunctions of a spectral problem defined by the unforced Maxwell's equations. One of these spectral problems stems from considering the permittivity as the eigenvalue, yielding the so-called generalized normal modes (GNM) [1, 2]. This approach entails several advantages over the alternative quasi-normal modes where the frequency is considered as the eigenvalue: GNM correspond to real stationary states (for a given real frequency), they don't suffer from an exponential growth at infinity, and are underpinned by a linear eigenvalue problem. However, to date, this approach is restricted to numerical experimentation only and lacks theoretical grounds. For instance, an important open question is whether the (generalized) eigenfunctions define a Riesz basis and therefore one can rigorously justify the modal expansion.

Owing to radiation conditions, the linear spectral problem underpinning GNM is non-self-adjoint, which is at the source of numerical and theoretical difficulties. Furthermore, the problem is non-standard in that its eigenvalues both diverge and accumulate at finite points. We are accordingly motivated to study the spectral properties of the GNM spectral problem. We focus, in particular, in GNM modes in 2D which are governed by the Helmholtz equation. We consider different scenarios, which are detailed in section 2. Our main results are the proof of the completeness of these modes in $H^1(D)$, where D is the domain occupied by the resonant cavity. We also show that they define a Riesz basis in the particular cases of the domains depicted in Fig. 1.

2 Problem formulation

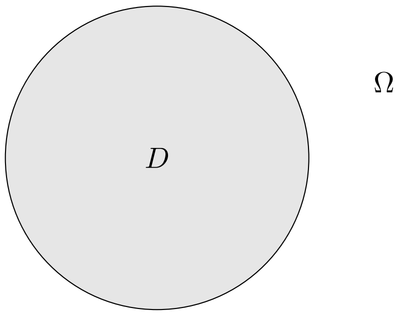
Consider a domain $\Omega \subset \mathbb{R}^2$, which extends (at least in one coordinate) to infinity. In particular we consider Ω to be a 2D waveguide or $\Omega = \mathbb{R}^2$. Consider also a compact and simply-connected domain $D \subset \Omega$ which we assume to be smooth over the transmission boundary with Ω. Fig. 1 shows two examples. We are interested in the following PDE spectral problem defined in Ω:

$$\begin{array}{c} \text{Find } \mathcal{E} \in \mathbb{C} \text{ and } u \in H^1_{\text{loc}}(\Omega) \\ \text{such that, for } \varepsilon = \chi(D)\,\mathcal{E} + \chi(\Omega\backslash D), \\ \nabla\left(\frac{1}{\varepsilon}\nabla u\right) + \omega^2 u = 0, \\ \lim_{x\to\infty} |x| \left(\frac{\partial}{\partial x} - i\omega\right) u(x) = 0 \end{array} \tag{1}$$

Here $\chi(\cdot)$ denotes the indicator function. If Ω is a waveguide, we further consider Neumann boundary conditions on the waveguide boundary.

3 Spectral analysis

We proceed by a variational approach and use a Dirichlet-to-Neumann mapping (which accounts for the radiation conditions) to reduce problem (1) to the study of a linear operator $A_\omega : H^1(D) \to H^1(D)$ with eigenvalues $1/\mathcal{E}$. The spectral properties of A_0 are well understood



(a) $D = \{(x, y);\, x^2 + y^2 < R\};\, \Omega = \mathbb{R}^2$.

(b) $D = \{(x, y); -a < x < 0, 0 < y < 1\}$; $\Omega = \{(x, y); \quad -a < x, 0 < y < 1\}$. In this case, we also consider Neumann boundary conditions on the upper walls and a Dirichlet boundary condition at $x = -a$.

Figure 1: Examples of domains D and Ω.

[3]. Furthermore, it can be shown that $A_\omega - A_0$ is compact. This yields the following result.

Lemma 1 *The spectrum of $A(\omega)$ is discrete with two accumulation points at $\mathcal{E} = \infty$ and $\mathcal{E} = -1$. Plus, all eigenvalues have a non-negative imaginary part.*

Each of this accumulation points is associated with a different family of modes. $\mathcal{E} = \infty$ is associated with bulk modes which can be used to describe resonances in high-index cavities; $\mathcal{E} = -1$ is associated with boundary modes which can be used to describe resonances in plasmonic resonators. This is illustrated in Fig. 2 and Fig. 3, which show some of the eigenfunctions for D of Fig. 1.(b). The numerical computation of these modes was carried out using the finite element software XLiFE++.

Using the above lemma along with the Riesz decomposition (or splitting) theorem we can show the existence of two subspaces $\mathcal{H}_\infty$ and $\mathcal{H}_{-1}$ such that $H^1(D) = \mathcal{H}_\infty \oplus \mathcal{H}_{-1}$, where A_ω restricted to $\mathcal{H}_i$ is invariant, with only one accumulation point ($\mathcal{E} = \infty$ or $\mathcal{E} = -1$), and of type Hilbert-Schmidt. Thus, using the theory of Hilbert-Schmidt operators yields the following result.

Theorem 2 *The generalized eigenfunctions of A_ω are complete in $H^1(D)$.*

To further show that the generalized eigenfunctions of A_ω form a Riesz basis of $H^1(D)$ it is sufficient to show that the application $\mathcal{T}(\alpha) := \sum_k \alpha_k u_k$, where $\{u_k\}$ is the sequence of generalized eigenfunctions of A_ω, defines an isomorphism of l^2 into $H^1(D)$. One can show that this property is directly linked to the asymptotic behaviour of the eigenvalues of A_ω. In the particular case of the configurations depicted in Fig. 1 such asymptotic behaviour can be estimated using separation of variables techniques, which yields the following result

Theorem 3 *The eigenvalues of the operator A_ω stemming from the domains depicted in Fig. 1 are not defective. Moreover, their eigenfunctions form a Riesz basis of $H^1(D)$.*

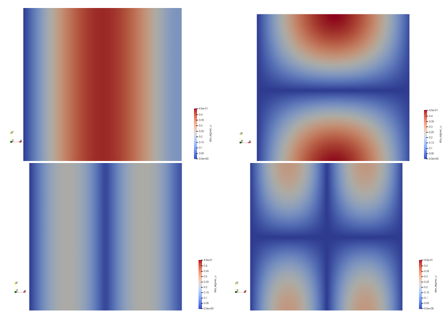


Figure 2: First four "bulk" eigenfunctions for D of Fig. 1.(b).

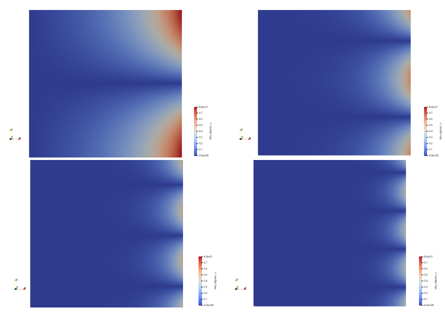


Figure 3: First four "plasmonic" eigenfunctions for D of Fig. 1.(b).

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References

[1] PY. Chen and D. Bergman, Y. Sivan, *Physical Review Applied* 11, 044018 (2019).

[2] C. Forestiere and G. Miano, *Physical Review B* 94, 201406(R) (2016).

[3] D. Grieser, F. Rüting, *J. Phys. A: Math. Theor.* 42 135204 (2009)