An iterative hybrid numerical-asymptotic boundary element method for high-frequency scattering by multiple screens

Oliver Phillips[1,*], **Stephen Langdon**[2], **Simon Chandler-Wilde**[1]

[1]Department of Mathematics and Statistics, University of Reading, Reading, UK

[2]Department of Mathematics, Brunel University London, London, UK

*Email: o.j.phillips@pgr.reading.ac.uk

Abstract

Standard Boundary Element Methods (BEM) for scattering problems, with piecewise polynomial approximation spaces, have a computational cost that grows with frequency. Recent Hybrid Numerical-Asymptotic (HNA) BEMs, with enriched approximation spaces consisting of the products of piecewise polynomials with carefully chosen oscillatory functions, have been shown to be effective in overcoming this limitation for a range of problems, focused on single convex scatterers or very specific non-convex or multiple scattering configurations. Here we present a novel HNA BEM approach to the problem of 2D scattering by a pair of screens in an arbitrary configuration, which we anticipate may serve as a building block towards algorithms for general multiple scattering problems with computational cost independent of frequency.

Keywords: High-frequency scattering, multiple scattering, BEM, hybrid numerical-asymptotic

1 Problem Statement

We consider the scattering of a plane wave $u^i(\mathbf{x}) := \mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$, where $\mathbf{d}$ is a unit vector in the direction of the plane wave and $k > 0$ is the wavenumber, by the union of two disjoint 1D screens, $\Gamma = \Gamma_1 \cup \Gamma_2$, in $D := \mathbb{R}^2 \setminus \overline{\Gamma}$, where $\overline{\Gamma}$ denotes the closure of Γ. The two screens can be in any orientation as long as they are not touching (e.g., Figure 1). The scattering problem we are looking to solve is to find $u \in C^2(D) \cap W^1_{\mathrm{loc}}(D)$ such that

$$\Delta u + k^2 u = 0 \quad \text{in} \quad D, \tag{1}$$
$$u = 0 \quad \text{on} \quad \Gamma, \tag{2}$$

and the scattered field $u^s = u - u^i$ satisfies the Sommerfeld radiation condition. By Green's 2$^{\text{nd}}$ identity (see, e.g., [2])

$$u(\mathbf{x}) = u^i(\mathbf{x}) - \frac{\mathrm{i}}{4}\int_\Gamma H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in D, \tag{3}$$

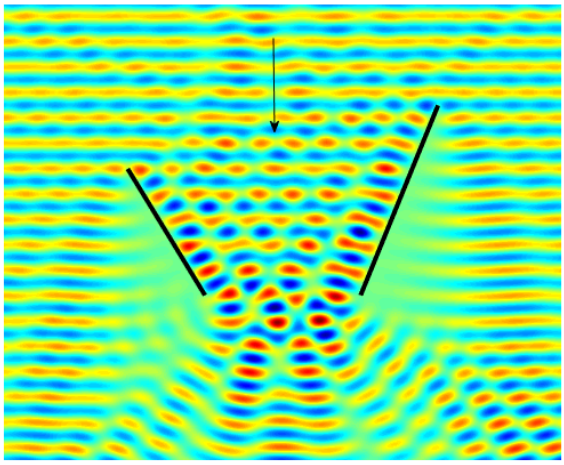

Figure 1: Re(u) in D, with Γ_1 on the left and Γ_2 on the right. The incident wave direction $\mathbf{d}$ is indicated by the arrow.

where $\phi \in \widetilde{H}^{-1/2}(\Gamma)$ is the jump in the normal derivative of u across Γ, and $H_0^{(1)}$ is the Hankel function of the first kind of order zero. Further, ϕ satisfies the boundary integral equation

$$\frac{\mathrm{i}}{4}\int_\Gamma H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) = u^i(\mathbf{x}), \quad \mathbf{x} \in \Gamma. \tag{4}$$

2 Multiple scattering iterative method

For ease of notation, define $\phi_j := \phi|_{\Gamma_j} \in \widetilde{H}^{-1/2}(\Gamma_j)$, and let $S_{\ell j} : \widetilde{H}^{-1/2}(\Gamma_j) \to H^{1/2}(\Gamma_\ell)$ be defined by

$$S_{\ell j}\psi(\mathbf{x}) := \frac{\mathrm{i}}{4}\int_{\Gamma_j} H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \tag{5}$$

for $\mathbf{x} \in \Gamma_\ell$, $\ell, j \in \{1,2\}$, and $\psi \in \widetilde{H}^{-1/2}(\Gamma_j)$. Equation (4) can then be written as

$$S_{11}\phi_1 + S_{12}\phi_2 = u^i|_{\Gamma_1}, \tag{6}$$
$$S_{21}\phi_1 + S_{22}\phi_2 = u^i|_{\Gamma_2}. \tag{7}$$

The first step in our iterative method is to ignore the effect of Γ_2 so (6) becomes

$$S_{11}\phi_1^{(0)} = u^i|_{\Gamma_1}, \tag{8}$$

where the 0 in the superscript refers to the number of the iteration considered. We next solve (7) for $\phi_2^{(1)}$, replacing ϕ_1 by $\phi_1^{(0)}$, thereby considering the first reflection from Γ_1 on Γ_2, solving

$$S_{22}\phi_2^{(1)} = u^i|_{\Gamma_2} - S_{21}\phi_1^{(0)}. \tag{9}$$

We then solve (6) with ϕ_2 replaced by $\phi_2^{(1)}$; in order to find the $2r^{th}$ order reflection on Γ_1 and $(2r+1)^{th}$ order reflection on Γ_2 we solve, for $r = 0, 1, 2, \ldots$, with $\phi_2^{(-1)} := 0$,

$$\begin{aligned} S_{11}\phi_1^{(2r)} &= u^i|_{\Gamma_1} - S_{12}\phi_2^{(2r-1)}, && (10)\\ S_{22}\phi_2^{(2r+1)} &= u^i|_{\Gamma_2} - S_{21}\phi_1^{(2r)}. && (11) \end{aligned}$$

3 High frequency approximation space

To solve (10) and (11) for a given r we propose to use an HNA BEM approximation space adapting that in [2]. The solution $\phi_1^{(2r)}$ to (10) can be decomposed as

$$\phi_1^{(2r)}(s) = \Psi_1^{(2r)}(s) + v_1^{+,2r}(s)\mathrm{e}^{\mathrm{i}ks} + v_1^{-,2r}(s)\mathrm{e}^{-\mathrm{i}ks}, \tag{12}$$

for $s \in [0, L_1]$, where L_1 is the length of Γ_1, and s denotes the distance from one of the end points. $\Psi_1^{(2r)}$ is the leading order physical optics high-frequency approximation, defined as twice the normal derivative of the field incident on Γ_1. Precisely, at this iteration,

$$\Psi_1^{(2r)} = 2\frac{\partial}{\partial n}\left(u^i - \mathcal{S}_2\widehat{\phi_2^{(2r-1)}}\right)\Big|_{\Gamma_1},$$

where, for $\psi \in \widetilde{H}^{-1/2}(\Gamma_j)$, $\mathcal{S}_j\psi \in C^2(D) \cap W^1_{\mathrm{loc}}(D)$ is given, for $j = 1, 2$, by

$$\mathcal{S}_j\psi(\mathbf{x}) := \frac{\mathrm{i}}{4}\int_{\Gamma_j} H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)\psi(\mathbf{y})ds(\mathbf{y}),\ \ \mathbf{x} \in D,$$

and $\widehat{\psi}(\mathbf{x}) := \psi(\mathbf{x})$ if a point source at $\mathbf{x}$ is incident on the same side of Γ_1 as u^i, otherwise $\widehat{\psi}(\mathbf{x}) := -\psi(\mathbf{x})$.

The term $\varphi_1^{(2r)}(s) := v_1^{+,2r}(s)\mathrm{e}^{\mathrm{i}ks} + v_1^{-,2r}(s)\mathrm{e}^{-\mathrm{i}ks}$ captures the diffraction from the corners. As in [2], it can be shown that the functions $v_1^{\pm,2r}$ in (12) are not oscillatory and hence can be approximated using standard piecewise polynomials with a number of degrees of freedom essentially independent of the wavenumber k. Therefore we can approximate $\varphi_1^{(2r)}$ by a sum of products of piecewise polynomials and $\mathrm{e}^{\pm\mathrm{i}ks}$ (our HNA BEM approximation space). Substituting (12) into (10) means we are solving, for $r = 0, 1, 2, \ldots$,

$$S_{11}\varphi_1^{(2r)} = u^i|_{\Gamma_1} - S_{12}\phi_2^{(2r-1)} - S_{11}\Psi_1^{(2r)}. \tag{13}$$

These equations can each be solved by either the Galerkin method, as in [2], or the least squares collocation method of [1], using the above HNA BEM approximation space, whichever method we choose.

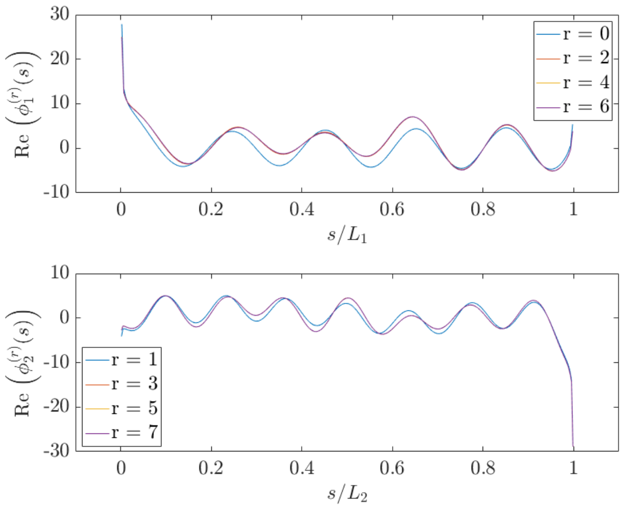

Figure 2: The iterates on Γ_1 (top) and Γ_2 (bottom) for the configuration of Figure 1, with $k = 5$.

4 Results

In this section we test the iterative component of the algorithm for the geometry in Figure 1. Solutions for various r can be seen in Figure 2, solving (10) and (11) by a conventional BEM. For this configuration we see convergence in very few iterations.

References

[1] A. Gibbs, D. P. Hewett, D. Huybrechs and E. Parolin, Fast hybrid numerical-asymptotic boundary element methods for high frequency screen and aperture problems based on least-squares collocation, *SN Partial Differential Equations and Applications*, **1** (2020) p.21.

[2] D. P. Hewett, S. Langdon and S. N. Chandler-Wilde, A frequency-independent boundary element method for scattering by two-dimensional screens and apertures, *IMA J. Numer. Anal.* **35** (2015) 1698-1728.