Analysis and FDTD simulation of a perfectly matched layer for the Drude metamaterial

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Abstract

In this talk, I'll present our recent work on analysis and application of a perfectly matched layer for the Drude metamaterial.

Keywords: Perfectly Matched Layer, metamaterial, FDTD scheme

1 Introduction

Since the Perfectly Matched Layer (PML) introduced for solving the three-dimensional (3D) time-dependent Maxwell's equations by Bérenger in 1994, many PML models have been proposed and studied further for Maxwell's equations (cf. [4, Ch.8] and references therein). The PML technique has also been extended to solve many other wave propagation problems, such as acoustics and elastodynamics (e.g. [1,2]).

In late 1990s, the so-called negative index metamaterials (NIMs) was manufactured successfully and immediately became a very hot research topic as evidenced by numerous papers and books published on metamaterials (e.g., [4] and references therein). In the same time, many studies of PMLs in NIMs have been carried out and found that the classical PMLs fail in NIMs. Soon, some stable PMLs were developed for metamaterials. Here we will present our recent work on developing and analyzing a FDTD scheme for the metamaterial PML model proposed by Bécache *et al.* [2].

2 The 2-D metamaterial PML model and its stability

A general 2-D Transverse Electric (TEz) metamaterial PML model wth 16 unknowns was developed in Bécache *et al.* Here we focus on the popular $\omega_e = \omega_m$ case whose governing equations can be written as follows (cf. [2, Eq.(48)]):

For any $(x, y, t) \in \Omega \times (0, T]$,

$$\partial_t E_x + \omega_e^2 J_x + \epsilon_0^{-1}\sigma_y E_x = \epsilon_0^{-1}\partial_y(H^x + H^y), \quad (1)$$
$$\partial_t J_x - E_x = 0, \quad (2)$$
$$\partial_t E_y + \omega_e^2 J_y + \epsilon_0^{-1}\sigma_x E_y = -\epsilon_0^{-1}\partial_x(H^x + H^y), \quad (3)$$
$$\partial_t J_y - E_y = 0, \quad (4)$$
$$\partial_t H^x + \omega_m^2 K^x + \mu_0^{-1}\sigma_y H^x = \mu_0^{-1}\partial_y E_x, \quad (5)$$
$$\partial_t K^x - H^x = 0, \quad (6)$$
$$\partial_t H^y + \omega_m^2 K^y + \mu_0^{-1}\sigma_x H^y = -\mu_0^{-1}\partial_x E_y, \quad (7)$$
$$\partial_t K^y - H^y = 0. \quad (8)$$

where ϵ_0 and μ_0 are the permittivity and permeability in free space, $\boldsymbol{E} = (E_x, E_y)$ and $H = H^x + H^y$ are the electric field and magnetic field (in split form) respectively, $\boldsymbol{J} = (J_x, J_y)$ and $\boldsymbol{K} = (K^x, K^y)$ are the auxiliary variables, $\sigma_x(x) \geq 0$ and $\sigma_y(y) \geq 0$ are the damping functions in the x and y directions, $\omega_e > 0$ and $\omega_m > 0$ are the electric and the magnetic plasma frequencies in the Drude model described by the following:

$$\epsilon(\omega) = \epsilon_0(1 - \frac{\omega_e^2}{\omega^2}), \quad \mu(\omega) = \mu_0(1 - \frac{\omega_m^2}{\omega^2}). \quad (9)$$

Here and in the rest of the paper, ω denotes the general wave frequency.

Since the PML is used in a rectangular domain outside the physical domain, we consider solving (1)-(8) in a rectangular domain $\Omega = [a, b] \times [c, d]$. To complete the model (1)-(8), we assume that the model problem is subject to the initial conditions

$$E_x(\boldsymbol{x}, 0) = E_{x0}(\boldsymbol{x}), \quad E_y(\boldsymbol{x}, 0) = E_{y0}(\boldsymbol{x}), \quad (10)$$
$$J_x(\boldsymbol{x}, 0) = J_{x0}(\boldsymbol{x}), \quad J_y(\boldsymbol{x}, 0) = J_{y0}(\boldsymbol{x}), \quad (11)$$
$$H^x(\boldsymbol{x}, 0) = H_{x0}(\boldsymbol{x}), H^y(\boldsymbol{x}, 0) = H_{y0}(\boldsymbol{x}), \quad (12)$$
$$K^x(\boldsymbol{x}, 0) = K_{x0}(\boldsymbol{x}), K^y(\boldsymbol{x}, 0) = K_{y0}(\boldsymbol{x}), \quad (13)$$

and the perfect conduct (PEC) boundary condition

$$E_x(x, y, t)|_{y=c,d} = 0, \quad E_y(x, y, t)|_{x=a,b} = 0, \quad (14)$$

where $E_{x0}, E_{y0}, J_{x0}, J_{y0}, H_{x0}, H_{y0}, K_{x0}$, and K_{y0} are some properly given functions.

In the rest of the paper, we denote the L^2 norm over Ω as $||\cdot|| := ||\cdot||_{L^2(\Omega)}$.

Theorem 1 *For the solution of (1)-(8), define the energy*

$$\mathcal{E}_1(t) = \frac{1}{2}\left[\epsilon_0(||E_x||^2 + ||E_y||^2) + \epsilon_0\omega_e^2(||J_x||^2 + ||J_y||^2) + \mu_0||H^x + H^y||^2 + \mu_0\omega_m^2||K^x + K^y||^2\right]. \quad (15)$$

Then for any nonnegative functions $\sigma_x(x)$ and $\sigma_y(y)$, we have

$$\frac{d}{dt}\mathcal{E}_1(t) + ||\sigma_y^{\frac{1}{2}} E_x||^2 + ||\sigma_x^{\frac{1}{2}} E_y||^2 + ||\sigma_y^{\frac{1}{2}} H^x||^2 \\ +||\sigma_x^{\frac{1}{2}} H^y||^2 + ((\sigma_x+\sigma_y)H^x, H^y) = 0. \quad (16)$$

When $\sigma_x = \sigma_y = \sigma \geq 0$ (i.e., a positive constant), the energy is decreasing:

$$\mathcal{E}_1(t) \leq \mathcal{E}_1(0), \quad \forall\ t \in [0,T]. \quad (17)$$

3 The FDTD scheme, stability analysis, and numerical simulation

To develop our difference scheme, we assume that the physical domain $\Omega = [a,b] \times [c,d]$ is partitioned by a uniform rectangular grid

$$a = x_0 < x_1 < \cdots < x_{N_x} = b, \\ c = y_0 < y_1 < \cdots < y_{N_y} = d,$$

and the time interval $[0,T]$ is partitioned into N_t uniform intervals by points $t_k = k\tau$, where $\tau = \frac{T}{N_t}, k = 0,1,\cdots,N_t$, grid points $x_i = ih_x, h_x = \frac{b-a}{N_x}, i = 0,1,\cdots,N_x$ in the x-direction, and grid points $y_j = jh_y, h_y = \frac{d-c}{N_y}, j = 0,1,\cdots,N_y$ in the y-direction. Our h_x and h_y can be different.

We introduce the following difference and averaging operators: For any discrete function $u_{i,j}^n$,

$$\delta_\tau u_{i,j}^{n+\frac{1}{2}} := \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\tau}, \quad \overline{u}_{i,j}^n = \frac{u_{i,j}^{n+\frac{1}{2}} + u_{i,j}^{n-\frac{1}{2}}}{2}, \\ \delta_x u_{i,j}^n := \frac{u_{i+\frac{1}{2},j}^n - u_{i-\frac{1}{2},j}^n}{h_x}, \quad \delta_y u_{i,j}^n := \frac{u_{i,j+\frac{1}{2}}^n - u_{i,j-\frac{1}{2}}^n}{h_y}.$$

We can develop the following FDTD scheme for solving the system of (1)-(8):

$$\delta_\tau E_{x,i+\frac{1}{2},j}^{n+\frac{1}{2}} + \omega_e^2 J_{x,i+\frac{1}{2},j}^{n+\frac{1}{2}} + \epsilon_0^{-1}\sigma_{y,j}\overline{E}_{x,i+\frac{1}{2},j}^{n+\frac{1}{2}} \\ = \epsilon_0^{-1}\delta_y(H^x+H^y)_{i+\frac{1}{2},j}^{n+\frac{1}{2}}, \quad (18)$$

$$\delta_\tau J_{x,i+\frac{1}{2},j}^n - E_{x,i+\frac{1}{2},j}^n = 0, \quad (19)$$

$$\delta_\tau E_{y,i,j+\frac{1}{2}}^{n+\frac{1}{2}} + \omega_e^2 J_{y,i,j+\frac{1}{2}}^{n+\frac{1}{2}} + \epsilon_0^{-1}\sigma_{x,i}\overline{E}_{y,i,j+\frac{1}{2}}^{n+\frac{1}{2}} \\ = -\epsilon_0^{-1}\delta_x(H^x+H^y)_{i,j+\frac{1}{2}}^{n+\frac{1}{2}}, \quad (20)$$

$$\delta_\tau J_{y,i,j+\frac{1}{2}}^n - E_{y,i,j+\frac{1}{2}}^n = 0, \quad (21)$$

$$\delta_\tau H_{i+\frac{1}{2},j+\frac{1}{2}}^{x,n+1} + \omega_m^2 K_{i+\frac{1}{2},j+\frac{1}{2}}^{x,n+1} + \mu_0^{-1}\sigma_{y,j+\frac{1}{2}}\overline{H}_{i+\frac{1}{2},j+\frac{1}{2}}^{x,n+1} \\ = \mu_0^{-1}\delta_y E_{x,i+\frac{1}{2},j+\frac{1}{2}}^{n+1}, \quad (22)$$

$$\delta_\tau K_{i+\frac{1}{2},j+\frac{1}{2}}^{x,n+\frac{1}{2}} - H_{i+\frac{1}{2},j+\frac{1}{2}}^{x,n+\frac{1}{2}} = 0, \quad (23)$$

$$\delta_\tau H_{i+\frac{1}{2},j+\frac{1}{2}}^{y,n+1} + \omega_m^2 K_{i+\frac{1}{2},j+\frac{1}{2}}^{y,n+1} + \mu_0^{-1}\sigma_{x,i+\frac{1}{2}}\overline{H}_{i+\frac{1}{2},j+\frac{1}{2}}^{y,n+1} \\ = -\mu_0^{-1}\delta_x E_{y,i+\frac{1}{2},j+\frac{1}{2}}^{n+1}, \quad (24)$$

$$\delta_\tau K_{i+\frac{1}{2},j+\frac{1}{2}}^{y,n+\frac{1}{2}} - H_{i+\frac{1}{2},j+\frac{1}{2}}^{y,n+\frac{1}{2}} = 0, \quad (25)$$

where we denote $\sigma_{y,j} = \sigma_y(y_j)$, and $E_{x,i+\frac{1}{2},j}^n \approx E_x(x_{i+\frac{1}{2}}, y_j, t_n)$, i.e., the approximate solution of E_x at point $(x_{i+\frac{1}{2}}, y_j, t_n)$. Similar notations are used for other variables. Many numerical tests are carried out, due to page limits, here we just present our simulation of a transmission problem between the vacuum and a Drude medium surrounded by Berenger's PML and the metamaterial PML respectively, originally proposed in [2].

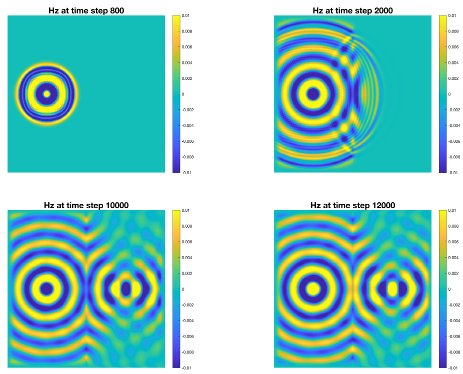

Figure 1: Snapshots of $H = H^x + H^y$ obtained by the scheme (18)-(25) with $\tau = 0.01$ at 800,2000,10000,12000 time steps.

References

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